CT ARML Team, 2022
Team Selection Test 1

1. The product of two positive integers is 132300. Compute the maximum possible value of the greatest common divisor of the two integers.
[Answer: 210]

2. The roots of the equation $32x^3 - 48x^2 - 26x + 21 = 0$ are in arithmetic progression. The difference between the largest and the smallest of the three roots is $\frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.
[Answer: 7]

3. There exists a digit *B* such that, for any digit *A*, the seven-digit number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{A}\,\underline{5}\,\underline{B}\,\underline{7}$ is not divisible by 11. Compute the digit *B*.
[Answer: 4]

4. A video game simulates the motion of a tiny, perfectly bouncy ball. The ball is projected from a vertex of an equilateral triangle of side length 1, and bounces in the interior of the triangle without stopping. Suppose now that the ball bounces off the sides of the triangle exactly 5 times before hitting a vertex of the triangle for the first time. Compute the square of the distance moved by the ball between projection and arrival at the vertex.
[Answer: 13]

5. Suppose that a point is randomly selected from the interior of a right triangle whose legs have length $2\sqrt{3}$ and 4. The probability that the distance of the point from its nearest vertex is less than 2 is $a + \pi\sqrt{b}$, where *a* and *b* are rational numbers. Find $\frac{1}{ab}$.
[Answer: 108]

6. Let $a_1, a_2, a_3, \ldots$ be an arithmetic sequence, let $b_1, b_2, b_3, \ldots$ be a geometric sequence, and let the sequence $c_1, c_2, c_3, \ldots$ be defined by $c_n = a_n + b_n$ for each positive integer *n*. If $c_1 = 1$, $c_2 = 4$, $c_3 = 15$, and $c_4 = 2$, what is the value of c_5 ?
[Answer: 61]

7. In the complex plane, the complex numbers 0, *z*, $\frac{1}{z}$, and $z + \frac{1}{z}$ form a parallelogram. If the area of the parallelogram is $\frac{35}{37}$, the smallest possible value of $\left|z + \frac{1}{z}\right|^2$ is $\frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.
[Answer: 87]

8. In square *PQRS* with diagonal of length 1, point *E* is on side $\overline{PQ}$ and point *F* is on side $\overline{QR}$, with $\text{m}\angle QRE = \text{m}\angle QPF = 30°$. Let *G* be the point of intersection of $\overline{RE}$ and $\overline{PF}$. The distance between the incenters of triangles *PGE* and *RGF* is $a - b\sqrt{c}$, where *a*, *b*, *c* are positive integers and *c* is not divisible by the square of any prime number. Find $a + b + c$.
[Answer: 9]

9. Let *R* be the region defined by the inequality $x^2 + y^2 \leq |x| + |y|$, and let the area of region *R* be *A*. Find the integer closest to 1000*A*.
[Answer: 5142]

10. Suppose that $6\tan^{-1} x + 4\tan^{-1}(3x) = \pi$. Then $x^2 = \frac{a-b\sqrt{c}}{d}$, where *a*, *b*, *c*, *d* are integers, $\gcd(a, b, d) = 1$, and *c* is not divisible by the square of any prime number. Find $a + b + c + d$.
[Answer: 59]