FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 6 Round 1
Arithmetic: Lines and angles

1.) {66,55,48}

2.) {54,53,59}

3.) {98,50,32}

Note: Solutions are provided for Form A only. All forms have similar solution methods.

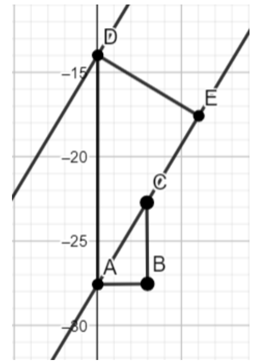
1.) If the measure of an angle in degrees is {11,6,19} more than {one-twelth, one-fifth, one-sixth} that of its supplement, what is the measure of its complement in degrees?

If x is the measure of the angle, then we can set up $x = 11 + \frac{1}{12}(180 - x)$, which we can solve to get $x = 24$, so the complement's measure in degrees is 66.

2.) $\angle FCM$ and $\angle LCD$ are vertical angles for which $m\angle FCM = (x^2)^{\circ}$ and $m\angle LCD = (xy + \{44,45,50\})^{\circ}$. If x and y are both integers and $y > 0$, find the sum of the greatest possible value of x and the greatest possible value of y.

Since the angles are vertical and therefore congruent, we have $x^2 = xy + 44$, which we can write as $x^2 - xy - 44 = 0$. If y is an integer, then this is a quadratic in x that has integer solutions and is therefore factorable. This gives ordered pairs (x, y) of $(44,43), (-1,43), (22,20), (-2,20), (11,7)$, and $(-4,7)$. Note that $x^2 \leq 180$ so the largest value of x is 11, and since the ordered pair $(-1,43)$ works, the largest value of y is 43, making the desired value 54.

3.) Line m has equation $5x - 3y = \{42{,}30{,}24\}$. Line n has equation $5x - 3y = K$ and is exactly $\{7{,}5{,}4\}$ units from line m. Find the product of all possible values of K.



Consult the diagram showing the two lines and two triangles drawn in. Note that $\Delta AED \sim \Delta CBA$, $AB = 3, BC = 5, ED = 7, AC = \sqrt{34}$, the coordinates of point D are $(0, -14)$ and the coordinates of point A are $\left(0, -\frac{K}{3}\right)$. We know that $\frac{AB}{AC} = \frac{DE}{AD}$, so $\frac{3}{\sqrt{34}} = \frac{7}{-14+\frac{K}{3}}$, or $-42 + K = 7\sqrt{34}$, so $K = 42 + 7\sqrt{34}$. We can see that the other possibility is $K = 42 - 7\sqrt{34}$, so the product is $42^2 - 34(7^2) = 98$.

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Match 6 Round 2 Algebra 1: Literal Equations

1.) $\{55,40,61\}$

2.) $\{28,84,62\}$

3.) $\{18,16,17\}$

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) If the equation $4x - y + 3z = 2(Ax + By)$, where A and B are constants, is solved for z, the result is $z = \{2,4,6\}x + \{7,3,5\}y$. Find the value of $3A + 4B$.

Isolating z gives $\frac{1}{3}\left((2A-4)x + (2B+1)y\right)$ and setting the coefficients for x and y equal to 2 and 7 respectively gives $A = 5$ and $B = 10$, so the desired quantity is $3(5) + 4(10) = 55$.

2.) If $x = \{3,5,4\}y + \sqrt{4y^2 + \{7,5,11\}}$ then there exist constants $a, b,$ and c such that $x^2 + axy + by^2 = c$. Find the value of $a + 4b + 2c$.

Setting $x - 3y = \sqrt{4y^2 + 7}$ and squaring both sides gives $x^2 - 6xy + 9y^2 = 4y^2 + 7$, or $x^2 - 6xy + 5y^2 = 7$, so the desired quantity is $(-6) + 4(5) + 2(7) = 28$.

3.) If the equation $x^2(y + 3M) = 4x(x^2 + Nx + 2) - 2(y - 5)$, where M and N are constants, is solved for y, the resulting equation can be written as $y = ax + b$ for some constants a and b. Find the value of $6M - 8N + \{2a + 4b, 4a + 2b, 3a + 3b\}$.

Expanding all products and grouping all terms with y on one side and factoring gives $y(x^2 + 2) = 4x^3 + (4N - 3M)x^2 + 8x + 10$. Because $y = ax + b$, that means that the right side can be written as $(ax + b)(x^2 + 2)$. We can then infer that $a = 4$ and $b = 5$, and expanding $(4x + 5)(x^2 + 2)$ we get $4x^3 + 5x^2 + 8x + 10$. Since $4N - 3M = 5$, then $6M - 8N = -10$, so the desired quantity is $-10 + 2(4) + 4(5) = 18$.

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Match 6 Round 3
Geometry: Solids and Volume

1.) $\{96,150,216\}$

2.) $\{43,13,28\}$

3.) $\{828,1938,258\}$

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) Consider a right cylinder whose height is equal the diameter of its bases. If this cylinder has a volume of $\{128\pi, 250\pi, 432\pi\}$, then its surface area is $k\pi$. What is the value of k?

Let r equal the radius of one base of the cylinder. This means the height is $2r$, and the volume is then $\pi r^2(2r) = 2\pi r^3 = 128\pi$, so $r = 4$. The surface area is then $2\pi(4^2) + 2\pi(4)(2 * 4) = 96\pi$, so $k = 96$.

2.) A cone is inscribed in a hemisphere and both share a circular base. If the surface area (including the base) of the cone is $\{15,5,10\}$ square units, then the surface area of the hemisphere is $a + b\sqrt{c}$ square units, where a, b, and c are integers and c has no perfect square factors larger than 1. Find the value of $2a + 3b - c$.

The surface area of a cone is $\pi r^2 + \pi r l$, where l is the slant height of the cone. Since the cone is inscribed in a hemisphere and shares its base, the height of the cone is equal to the radius of the base, so $l = r\sqrt{2}$. This makes the surface area $\pi r^2\left(1 + \sqrt{2}\right)$. The surface area of the hemisphere is $3\pi r^2$, so in this case it will be $3\left(\frac{15}{1+\sqrt{2}}\right)$, or $45\left(\sqrt{2} - 1\right)$, or $-45 + 45\sqrt{2}$, making the desired quantity $2(-45) + 3(45) - 2 = 43$.

3.) A cube is inscribed a hemisphere of radius {9,12,6} units such that the center of the base of the cube rests on the center of the base of the hemisphere. The volume inside the hemisphere but outside the cube can be written in the form of $A\pi - B\sqrt{C}$ units, where A, B, and C are positive integers with C having no perfect square factors greater than 1. Find the value of $A + 2B + 3C$.

The distance from one of the top four vertices of the cube to the center of the bottom face of the cube will be equal to the radius of the hemisphere. If s is the length of one side of the cube, then $r^2 = s^2 + \left(\frac{\sqrt{2}}{2}s\right)^2 = \frac{3}{2}s^2$, so $r = \frac{\sqrt{6}}{2}s$ and $s = \frac{\sqrt{6}}{3}r$. If $r = 9$, then $s = 3\sqrt{6}$, and the volume inside the hemisphere but outside the cube is $\frac{2}{3}\pi(9^3) - \left(3\sqrt{6}\right)^3$, or $486\pi - 162\sqrt{6}$. Therefore, the desired quantity is $486 + 2(162) + 3(6) = 828$.

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Match 6 Round 4
Algebra 2: Radical expressions and equations

1.) $\{3,2,4\}$

2.) $\{6972,5852,4970\}$

3.) $\{812,870,930\}$

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) What is the extraneous solution to $\sqrt{x-\{2,1,3\}}+\{4,3,5\}=x$?

Subtracting 4 from both sides gives $\sqrt{x-2}=x-4$, and squaring both sides produces $x-2=x^2-8x+16$, which in standard form makes $x^2-9x+18$. This factors and produces solutions of $x=6$ and $x=3$, but since $x=3$ does not satisfy the original equation, it is extraneous and therefore the desired result.

2.) What is the largest value of $n \le \{7000,6000,5000\}$ such that the expression $\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}$ evaluates to an integer?

Setting $x=\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}$ and recognizing this is equivalent to $x=\sqrt{n+x}$ leads to the equation $x^2-x-n=0$. If this has integer solution for x, it would be factorable into $(x-k)(x+k-1)=0$, where $k(k-1)=n$ and the desired non-extraneous solution would be $x=k$. The largest number $n<7000$ that is the product of two consecutive integers is $6972=(84)(83)$, so $n=6972$.

3.) If $a_0 = 0$ and $a_n = a_{n-1} + 1 + \sqrt{4a_{n-1} + 1}$ for $n > 0$, what is the value of $\{a_{28}, a_{29}, a_{30}\}$?

Using the recursive formula tells us that $a_1 = 2$, $a_2 = 6$, and $a_3 = 12$. Note that $a_1 = (2)(1)$, $a_2 = (3)(2)$, and $a_3 = (4)(3)$, or more generally $a_n = (n + 1)(n)$. Therefore $a_{28} = (29)(28) = 812$.

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Match 6 Round 5
Precalculus: Polynomials
and Advanced Factoring

1.) $\{416,224,266\}$

2.) $\{660,390,273\}$

3.) $\{27,32,37\}$

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) The function $f(x) = x^2 + ax + 24$ for some constant a has one zero of $x = \{-2, -4, -3\}$. What is $f(a)$?

Since $(-2)^2 - 2a + 24 = 0$, we know that $a = 14$. This means $f(14) = (14)^2 + 14(14) + 24 = 416$.

2.) A polynomial equation $x^4 + ax^2 + bx = c$ where a, b, and c are integers has a solution set that includes $x = \{4 + 2i, 1 + 5i, 2 + 3i\}$ and $x = 3$. What is the value of c?

Because the there is no cubic term, we know that the roots of the polynomial equation must add to 0, and since all the coefficients are integers, it follows that one additional root must be $4 - 2i$. The sum of the three known roots is then $4 + 2i + 4 - 2i + 3 = 11$, so the remaining root must be $x = -11$. Since c will end up being the opposite of the product of the roots, $c = -(-11)(3)(4 + 2i)(4 - 2i) = -(-33)(20) = 660$.

3.) The polynomial $x^4 + ax^3 + bx^2 + cx - 4$, where a, b, and c are integer coefficients, factors into $(x^2 + 4)(x^2 + (c - 3a)x + a - 2b)$. Find the value of $\{2a - b + c, 3a - b + c, 4a - b + c\}$.

Multiplying out the two factors gives $x^4 + (c - 3a)x^3 + (a - 2b + 4)x^2 + 4(c - 3a)x + 4(a - 2b)$. Comparing to the first form of the polynomial gives the equations $c - 3a = a$, $a - 2b + 4 = b$, $4(c - 3a) = c$, and $4(a - 2b) = -4$. We know from the last of these that $a - 2b = -1$, and since $a - 3b = -4$, we know $b = 3$ and subsequently $a = 5$. We also know that $c = 4a$, so $c = 20$. Therefore, the desired quantity is $2(5) - 3 + 20 = 27$.

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Match 6 Round 6
Miscellaneous: Counting and Probability

1.) $\{8,12,15\}$

2.) $\{69,82,95\}$

3.) $\{36000,4320,576\}$

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) A bin contains M marbles, 5 of which are red. The probability of selecting two red marbles randomly from the bin is $\left\{\frac{5}{14},\frac{5}{33},\frac{2}{21}\right\}$. What is the value of M?

The probability of choosing 2 red marbles without replacement from the bin of originally M marbles can be written as $\frac{5}{M} * \frac{4}{M-1}$, and setting equal to $\frac{5}{14}$ gives $M(M-1) = 56$, which gives $M = 8$.

2.) A basketball player makes $k\%$ of his free-throws. He practices by throwing N free-throws. If each free-throw is independent, the probability that he will make at least n of the shots is found by computing $1 - \binom{\{10,11,12\}}{0}\{.35,.45,.55\}^0\{.65^{10},.55^{11},.45^{12}\} - \binom{\{10,11,12\}}{1}\{.35,.45,.55\}^1\{.65^9,.55^{10},.45^{11}\}$. Find the value of $2n + 3N + k$.

Since the binomial form shows choosing cases from 10 possible and the total of the exponents in each term is 10, there are 10 total free-throws. Making at least

n is the complement of making up to $n - 1$, so the computation subtracts off the case of making 0 or 1 free-throws. This tells us that $n = 2$ and $k = 35$, so the desired quantity is $2(2) + 3(10) + 35 = 69$.

3.) A group of {8,7,6} students is standing in a row for a picture. The photographer does not want the three tallest students grouped together. Two of them *can* be next to each other as long as the third is not next to either one. How many ways can the students be arranged for the picture? (Assume left-to-right order matters.)

The easiest way to count suitable cases is to count the cases that fail (i.e. the three students are standing together) and subtracting from the total number of possible orders. We can assume the three students are grouped together as a "single student" which makes 6! arrangements, but to account for the 3! possible arrangements of the three students, we have a total of $3!\,6! = 4320$ arrangements. Subtracting from the 8! total possible gives 36000 possible arrangements.

Team Round

FAIRFIELD COUNTY MATH LEAGUE 2021-2022 Match 6 Team Round

1.) 2022	4.) 94
2.) 1234	5.) 82
3.) 79	6.) 65

1.) A convex 20-gon has the following properties: it has A acute angles each with a measure of B degrees, C obtuse angles each with a measure of D degrees, and no right angles. Angles with measures B and D are supplementary. If $A, B, C,$ and D are positive integers, find the value of $8A + 3B + 6C + 11D$.

Since $A = 20 - C$ and $D = 180 - B$, we have $(20 - C)(B) + (C)(180 - B) = 3240$. This means $20B + 180C - 2BC = 3240$. Solving for B in terms of C gives $B = \frac{3240-180C}{20-2C} = \frac{1620-90C}{10-C}$. Since $1 \leq C \leq 19$ and $1 \leq B \leq 89$, the only values that work are $C = 19$, $B = 10$, $A = 1$, and $D = 170$, so the desired quantity is $8(1) + 3(10) + 6(19) + 11(170) = 2022$.

2.) If $z = 5x - \frac{2}{y}$ is a solution to the equation $y^2(z - Ax)(z + Ax) + Bxy = C$ for positive constants $A, B,$ and C, find the value of $40A^2 + 11B + \frac{7}{2}C$.

Expanding and solving for z^2 in the larger equation gives $z^2 = \frac{C}{y^2} - \frac{Bx}{y} + A^2x^2 = \frac{C-Bxy+A^2x^2y^2}{y^2}$. Squaring the first equation gives $z^2 = \frac{4-20xy+25x^2y^2}{y^2}$, so $A^2 = 25$, $B = 20$, and $C = 4$, making the desired quantity $40(25) + 11(20) + \frac{7}{2}(4) = 1234$.

3.) A cylinder with a height of 10 inches and a base radius of 12 inches is sliced by a plane along the furthest distance from its top base to its bottom. (Note: the plane is diagonal such that the plane intersects each base at a single point, dividing the original cylinder into two congruent solids.) The ratio of the surface area of one of the two resulting congruent solids to that of the original cylinder can be written in simplest form as $\frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$. (Note: the area of an ellipse is $\pi r_1 r_2$ where r_1 and r_2 are the lengths of the semi-major and semi-minor axes.)

The surface area of the resulting solid contains one of the original bases, exactly half of the original cylinder's lateral surface area, and a new surface which takes the shape of an ellipse with a minor radius of 12 (the radius of the base) and a major radius equaling half the maximum length from the top of the new surface to the base. This length is $\frac{1}{2}\sqrt{10^2 + 24^2} = 13$. Therefore, the surface area of the new solid is $\pi 12^2 + \pi(10)(12) + \pi(12)(13) = 420\pi$, and since the surface area of the original cylinder was 528π, making the ratio in simplest form $\frac{35}{44}$. Therefore, the desired quantity is $35 + 44 = 79$.

4.) Find the sum of all values of x such that there exists an ordered pair (x, y) where x and y are positive integers and $\sqrt{x+y} + \sqrt{x-y} = 8$.

Subtracting $\sqrt{x-y}$ and squaring both sides yields $x + y = 64 + x - y - 16\sqrt{x-y}$. Isolating the radical produces $\sqrt{x-y} = 4 - \frac{1}{8}y$, and squaring again makes $x - y = 16 - y + \frac{1}{64}y^2$, or $x = 16 + \frac{1}{64}y^2$. Since both x and y must be positive integers, it follows that only multiples of 8 will be suitable as values for y. This gives ordered pairs of $(17,8), (20,16), (25,24)$, and $(32,32)$ (we can see that increasing y further makes both numbers too large to work in the original equation and therefore makes extraneous solutions). Therefore, the desired value is $17 + 20 + 25 + 32 = 94$.

5.) Consider the polynomial $f(x) = x^3 + 5x^2 + ax + b$, where a and b are real constant coefficients and $b < 0$. The sum of the squares of the zeros of $f(x)$ is 13 and $f(b) = 31b$. What is the value of $f(3)$?

Let $u, v,$ and w be the zeros of $f(x)$. We know that $u + v + w = -5$, and therefore $(u + v + w)^2 = u^2 + v^2 + w^2 + 2uv + 2uw + 2vw = 25$. Since the sum of the squared zeros is 13, we know $uv + uw + vw = 6$. By Viète's formulas this is also the value of a. Now $f(b) = b^3 + 5b^2 + 6b + b = 31b$, so $b^3 + 5b^2 - 24b = 0$. This factors to make $b(b + 8)(b - 3) = 0$, and since $b < 0$, we know $b = -8$. Therefore $f(3) = 3^3 + 5 * 3^2 + 6 * 3 - 8 = 82$.

6.) In the town of Sunnyville, days are either sunny or rainy. If it is sunny today, there is a 72% chance it will be sunny tomorrow. If it is rainy today, there is a 48% chance it will be rainy tomorrow. The overall proportion of sunny days over time in Sunnyville is p. Find $100p$.

There are many approaches to this problem. One approach is to find the probability p_{s2}, the probability it will be sunny tomorrow given the probability it is sunny today, p_{s1}. This comes out to $p_{s2} = .72p_{s1} + (1 - .48)(1 - p_{s1}) = .52 + .2p_{s1}$. From here the easiest method is to solve for the "stable state", that is when $p_{s2} = p_{s1} = p$, or $p = .52 + .2p$, making $p = .65$. Therefore the desired value is 65.