FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 5 Round 1 Arithmetic: Fractions and Exponents

1.) $\{78,18,10\}$

2.) $\{18,28,20\}$

3.) $\{58,59,60\}$

Note: Solutions provided are for Form A only. All forms have similar solution methods.

1.) How many integers x have the property that $x^{\{2,4,6\}}$ is between $\{1000,100000,10000000\}$ and $\{5000,500000,50000000\}$?

If $1000 \leq x^2 \leq 5000$, then $10\sqrt{10} \leq |x| \leq 50\sqrt{2}$, and since $10\sqrt{10} \approx 31.6$ and $50\sqrt{2} \approx 70.7$, there are 39 positive and 39 negative integers that satisfy the inequality, so the total number is 78.

2.) Given the least integer value of $n > \{4,5,6\}$ such that the average (arithmetic mean) of $\frac{1}{\{4,5,6\}}$ and $\frac{1}{n}$ is of the form $\frac{1}{b}$ where b is an integer, what is the value of $n + b$?

Setting $\frac{1}{2}\left(\frac{1}{4} + \frac{1}{n}\right) = \frac{1}{b}$ and solving for b in terms of n gives $b = \frac{8n}{4+n}$. By inspection the smallest value of n that produces an integer value of b is $n = 12$, which yields $b = 6$, so $n + b = 18$.

3.) If a and b are the largest positive two-digit integers possible such that $\frac{32^a}{\left\{\sqrt[4]{16},\sqrt[3]{64},\sqrt[3]{512}\right\}} = 8^{\frac{4}{3}b}$, find $2a - b$.

Multiplying both sides by $2\left(\sqrt[4]{16}\right)$ yields $32^{a} = 2 * 8^{\frac{4}{3}b}$, and if both sides are written in terms of base 2 we get $2^{5a} = 2^{4b+1}$, so $5a = 4b + 1$, or $5a - 4b = 1$. By inspection, $a = 1$ and $b = 1$ satisfy the equation, and the next integer ordered pair to do so can be found by increasing a by 4 and b by 5. We can increase b by 5 nineteen times and still be less than 100, so $a = 1 + 19(4) = 77$ and $b = 1 + 19(5) = 96$, so $2(77) - 96 = 58$.

FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 5 Round 2
Algebra 1: Rational Expressions & Equations

1.) $\{42,38,30\}$

2.) $\{2159,2061,1963\}$

3.) $\{36,50,45\}$

Note: Solutions provided are for Form A only. All forms have similar solution methods.

1.) The ratio of the sum of $\frac{\{9,7,3\}}{x}$ and $\frac{5}{x-1}$ to the difference between $\frac{4}{x}$ and $\frac{6}{x^2-1}$ can be written in simplest form as $\frac{Ax^2+Bx+C}{Dx^2+Ex+F}$ where $A>0$ and all coefficients have no common factors larger than 1. Find the value of $|A|+|B|+|C|+|D|+|E|+|F|$.

Setting up the fractional expression $\frac{\frac{9}{x}+\frac{5}{x-1}}{\frac{4}{x}-\frac{6}{x^2-1}}$ and multiplying both the numerator and denominator by $x(x^2-1)$ yields $\frac{9(x^2-1)+5x(x+1)}{4(x^2-1)-6}$, which can be written as $\frac{14x^2+5x-9}{4x^2-10}$, making the desired quantity $14+5+9+4+10=42$.

2.) There are two values of k such that the equation $\frac{5x-17}{x-3}=18x+k$ has exactly one real solution for x. If the sum of the two values is s and the product is p, find $\{p+s,p+2s,p+3s\}$.

Multiplying both sides by $x-3$ yields $5x-17=(18x+k)(x-3)$, which can be expanded and written in standard form in terms of x as $18x^2+(k-59)x+17-3k=0$. The discriminant of this equation in terms of k is

$(k-59)^2 - 4(18)(17-3k)$, or $k^2 + 98k + 2257$. This makes the sum of the desired values of k (the ones that make this expression zero) -98 and the product 2257, so $2257 - 98 = 2159$.

3.) Ivan and Sandra both commute to the office. Ivan's commute is 45 miles and Sandra's commute is 36 miles. One day they leave their houses at the same time. Sandra's average speed is {12,20,15} miles per hour less than Ivan's average speed and she arrives at the office $\{15,18,12\}$ minutes after Ivan. What was Ivan's average speed for his commute to work in miles per hour?

Let r be Ivan's average speed in miles per hour. Knowing that 15 minutes is $\frac{1}{4}$ hours, we can set up $\frac{45}{r} = \frac{36}{r-12} - \frac{1}{4}$ to set Ivan's commute time equal to Sandra's commute time. Multiplying everything by $4r(r-12)$ yields $45(4)(r-12) = 36(4)(r) - r(r-12)$, which expanded and written in standard form in terms of r yields $r^2 + 24r - 2160 = 0$. Factoring yields $(r-36)(r+60) = 0$, and since $r = -60$ is extraneous, the desired answer is $r = 36$.

FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 5 Round 3 Geometry: Circles

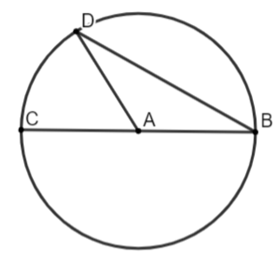
1.) {40,20,30}

2.) {3969,2916,2025}

3.) {36,64,100}

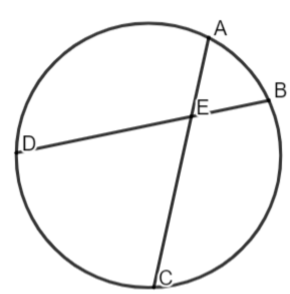
Note: Figures may not be drawn to scale. Solutions provided are for Form A only. All forms have similar solution methods.

1.) Consider the diagram. The circle has center A and diameter $\overline{BC}$. If the ratio of $m\widehat{BD}$ to $m\widehat{CD}$ is $\{5:4, 7:2, 6:3\}$, find $m\angle DBC$ in degrees.



Since the parts of the ratio add to a total of 9, that gives $m\widehat{BD} = 100^{\circ}$ and $m\widehat{CD} = 80^{\circ}$. Since $\widehat{CD}$ subtends the inscribed angle $\angle DBC$, it follows that $m\angle DBC = 40^{\circ}$.

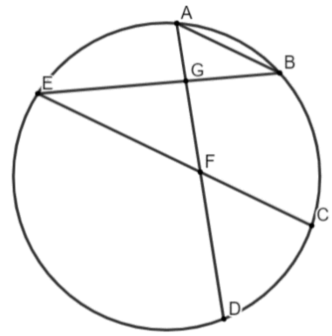
2.) Consider the diagram where chords $\overline{AC}$ and $\overline{BD}$ intersect at point E. $m\widehat{CD} = 70^{\circ}$, $m\angle AEB = 45^{\circ}$, and the length of $\widehat{AB}$ is $\{7,6,5\}\pi$ units. If the area of the circle is $k\pi$, find the value of k.



Because $m\angle AEB = \frac{m\widehat{CD} + m\widehat{AB}}{2}$, we have $m\widehat{AB} = 20^{\circ}$.

Since this is $\frac{1}{18}$ of the circumference, the length of the circumference is 126π, making the radius 63 units. The area is then $(63)^2\pi = 3969\pi$, so $k = 3969$.

3.) Consider the diagram. Chords $\overline{AD}$ and $\overline{BE}$ intersect at point G, chords $\overline{AD}$ and $\overline{EC}$ intersect at point F, and $\overline{AB}||\overline{EC}$. If $AG = \{6,8,10\}$, $GF = \{9,12,15\}$, $BG = x$ and $CF = y$, then there exist constants a and b such that $y = ax^2 - b$ for any value of x. Find the value of $\frac{b}{a}$.



Let $FD = z$ and $AB = k$. Because $\Delta ABG \sim \Delta FEG$ (due to the parallel chords), it follows that $EF = 1.5k$ and $GE = 1.5x$. We then have two relationships based on intersecting chords: $(AG)(GD) = (BG)(GE)$ and $(AF)(FD) = (EF)(FC)$, yielding $(6)(9 + z) = x(1.5x)$ and $(15)(z) = (1.5k)(y)$ respectively. Since z is in both equations, we can isolate z and set the corresponding relationships equal. Since $z = \frac{1}{4}x^2 - 9$ from the first and $z = \frac{1}{10}ky$ from the second, we have $\frac{1}{4}x^2 - 9 = \frac{1}{10}ky$, or $y = \frac{5}{2k}x^2 - \frac{90}{k}$, making the desired ratio $\frac{180k}{5k} = 36$.

FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 5 Round 4 Algebra 2: Quadratic Equations and Complex Numbers

1.) {69,49,45}

2.) {144,256,324}

3.) {108,48,192}

Note: Solutions provided are for Form A only. All forms have similar solution methods.

1.) If the quadratic equation $ax^2 + bx + c = 0$, where $a, b,$ and c are integers with no common factors greater than 1 and $a > 0$, has one solution of $\left\{3 + \frac{1}{2}i, 2 + \frac{3}{2}i, 1 + \frac{5}{2}i\right\}$, find the value of $2a - b + c$.

Since the equation has integer coefficients, it follows that the other solution must be $3 - \frac{1}{2}i$. This makes the sum of the solutions 6 and the product $\frac{37}{4}$. Therefore the equation would be $x^2 - 6x + \frac{37}{4} = 0$, and then multiplying by 4 so that every coefficient is an integer gives $4x^2 - 24x + 37 = 0$, so $2(4) - (-24) + 37 = 69$.

2.) The three functions $f(z)$, $g(z)$, and $h(z)$ have the properties that each shares exactly one complex zero with each of the other two, but all three have no complex zeros in common. If $f(z) = z^2 - \{12,16,18\}z + \{45,73,90\}$, $g(z) = z^2 - \{27 - 36i, 55 - 48i, 72 - 54i\}$ and $h(z)$ is a quadratic function with a leading coefficient of 1, what is the value of the discriminant of $h(z)$?

We can use the quadratic formula to determine that the zeros of $f(z)$ are $6 + 3i$ and $6 - 3i$. For $g(z)$ the zeros must be opposite since the first degree coefficient is zero. Writing it as $g(z) = z^2 - (27 + 36i)$ which is $g(z) = z^2 - (6 + 3i)^2$ shows the zeros to be $6 + 3i$ and $-6 - 3i$. This means that the zeros of $h(z)$ must be $6 - 3i$ and $-6 - 3i$. This makes the equation for $h(z) = z^2 + 6iz - 45$, so the discriminant is $(6i)^2 - 4(-45) = 144$. Alternatively, we can reason that $-3i \pm 6 = \frac{-6i \pm \sqrt{144}}{2}$.

3.) Let $f(z) = 2z^2 - (12 + 8i)z + 15 + 24i$. For a particular complex number $k = a + bi$ where a and b are real, it is known that $f(k)$ is real and $|k| = \{2\sqrt{10}, 2\sqrt{5}, 2\sqrt{17}\}$. Find the magnitude of the product of all possible values of a.

Putting $f(z)$ into vertex form by completing the square shows $f(z) = 2(z^2 - (6 + 4i)z + (3 + 2i)^2) + 15 + 24i - 2(3 + 2i)^2 = 2\left(z - (3 + 2i)\right)^2 + 5$. In order for $f(a + bi)$ to be real, either the real or imaginary term of $a + bi - (3 + 2i)$ must be zero before the result is squared. This makes $a = 3$ or $b = 2$. Since $|k| = 2\sqrt{10}$, it follows also that $a^2 + b^2 = 40$. This gives possible values of $3 \pm \sqrt{31}i$ or $\pm 6 + 2i$. With three different possible values of a, the desired result is $|(3)(6)(-6)| = 108$.

FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 5 Round 5 Precalculus: Trigonometric Equations

1.) {37,53,62}

2.) {18,32,8}

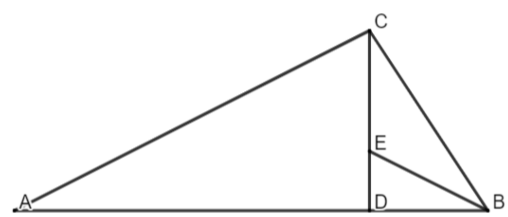
3.) {1,7,8}

Note: Solutions provided are for Form A only. All forms have similar solution methods.

1.) In right triangle FEB with right angle E, if $FE = \{2,3,4\}BE$, find $m\angle FBE - m\angle BFE$ to the nearest whole degree. (Do not round angle measures until the last step.)

Since $m\angle FBE = \arctan(2)$ and $m\angle BFE = \arctan\left(\frac{1}{2}\right)$, the desired result is $\arctan(2) - \arctan\left(\frac{1}{2}\right) \approx 36.8699^{\circ} \approx 37^{\circ}$.

2.) Consider the diagram (not drawn to scale). $CD = 1, AD = \{7,9,5\}, BD = \{5,7,3\}$, $\angle CAB \cong \angle CBE$, and $\overline{CD}$ is an altitude of the triangle. If $m\angle EBD = \theta$, find $\cot(\theta)$.



Note that $\tan(m\angle CAB) = \frac{1}{7}$ and $\tan(m\angle CBA) = \tan(m\angle CBE + \theta) = \frac{1}{5}$. Since $\angle CAB \cong \angle CBE$, it follows that $\frac{1}{5} = \frac{\frac{1}{7}+\tan(\theta)}{1-\frac{1}{7}\tan(\theta)}$, so $\frac{36}{35}\tan(\theta) = \frac{2}{35}$, making $\tan(\theta) = \frac{1}{18}$, so $\cot(\theta) = 18$.

3.) If $\cos\left(x+\frac{\pi}{4}\right) = y$, then there exist constants a and b such that $\sin(2x) = ay^2 + b$ for all x and y. Find the value of $\{2a + 5b, b - 3a, a + 10b\}$.

Expanding the cosine expression yields $\cos(x)\cos\left(\frac{\pi}{4}\right) - \sin(x)\sin\left(\frac{\pi}{4}\right) = y$, which means $\cos(x) - \sin(x) = \sqrt{2}y$. Squaring both sides yields $\cos^2(x) - 2\sin(x)\cos(x) + \sin^2(x) = 2y^2$, or $2\sin(x)\cos(x) = \sin(2x) = -2y^2 + 1$, so the desired result is $2(-2) + 5(1) = 1$.

FAIRFIELD COUNTY MATH LEAGUE 2021-22

Match 5 Round 6
Miscellaneous: Sequences and Series

1.) $\{35,53,44\}$

2.) $\{40,32,24\}$

3.) $\{35,39,48\}$

Note: Solutions provided are for Form A only. All forms have similar solution methods.

1.) If the first term in an arithmetic sequence is 8 and the sum of the fourth and fifth terms is $\{37,51,44\}$, what is the tenth term of the sequence?

If d is the common difference of the sequence, then the fourth term would be $8+3d$ and the fifth term would be $8+4d$, so $16+7d=37$, making $d=3$, so the tenth term is $8+9(3)=35$.

2.) For a particular infinite geometric series, if the sum of the first 1011 terms is half the infinite sum of the series and the difference between the first term and 2023rd term is $\{30,24,18\}$, what is the first term of the series?

Letting the first term be a and the ratio between terms be r, we have $\frac{a(1-r^{1011})}{1-r}=\frac{1}{2}*\frac{a}{1-r}$, which simplifies to $1-r^{1011}=\frac{1}{2}$. This also means that $r^{1011}=\frac{1}{2}$. From the second relationship, we have $a-ar^{2022}=30$, which factors to make $a(1-r^{1011})(1+r^{1011})=30$, or $a\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=30$, so $a=40$.

3.) A particular arithmetic sequence has a third term of $\{19,20,15\}$ and a fifth term of $\{35,32,23\}$. The sum of the first n terms is S. If $1000<S<10000$, how many possible values of n are there?

Let the first term be a and the difference between terms be d. Since $a + 2d = 19$ and $a + 4d = 35$, we can find that $a = 3$ and $d = 8$. From here, we know that the sum of the first n terms is $S = \sum_{k=0}^{n-1}(a + kd) = na + \frac{n(n-1)}{2}d$, so in this case the sum is $S = 3n + 4n(n - 1) = 4n^2 - n$. Solving the quadratic for each bound or checking with technology shows $16 \leq n \leq 50$ for integer values of n, so there are a total of 35 possible values of n.

Team Round

FAIRFIELD COUNTY MATH LEAGUE 2021-2022 Match 5 Team Round

1.) 12

2.) 30

3.) 18

4.) 2700

5.) 6

6.) 130

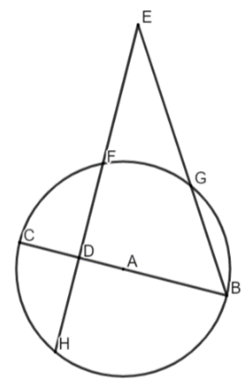
1.) If $5^{a-b+c} = 10$ and $5^{a+b-c} = 30$, what is the value of 25^{a-1}?

Multiplying the first two equations together makes $5^{2a} = 300$, or $25^a = 300$. Since $300 = 25 * 12$, we have $25^a = 25 * 12$ or $25^{a-1} = 12$.

2.) A total cost of \$$N$ is to be split evenly among $P > 4$ people. If 4 people refuse to pay, then each remaining person must pay \$10 more than what they would have paid if everyone contributed. If N is a positive integer less than 10000, how many possible values of N exist?

The situation can be modeled by $\frac{N}{P} + 10 = \frac{N}{P-4}$. If we solve for N, we get $N = \frac{5}{2}P^2 - 10P$. In order for N to be an integer, P must be even. This means the smallest possible value for P is 6, and solving the quadratic for $N = 10000$ or inspecting with technology shows the largest value of P that works is 64. Therefore we need the number of even integers from 6 to 64 inclusive which is 30.

3.) Consider the diagram (not drawn to scale). The circle has center A and diameter $\overline{BC}$, and $\overline{EH} \perp \overline{BC}$. If the radius of the circle is 13, $AD = 5$, and $GE = 8$, find the value of BG.



One way this can be solved is by drawing $\overline{CG}$, and then using the fact that $\Delta CGB \sim \Delta EDB$. Letting $BG = x$ and setting up $\frac{BC}{BG} = \frac{BE}{BD}$ gives $\frac{26}{x} = \frac{x+8}{18}$, which yields the equation $x^2 + 8x - 468 = 0$, whose non-extraneous solution is $x = 18$.

4.) For a particular geometric sequence of complex numbers, $a_5 = 3 + i$ and $a_{12} = 25 + 5\sqrt{11}i$. Find the value of $|a_{26}|$.

Note that $|a_5| = \sqrt{10}$ and $|a_{12}| = 30$. This means that $|r^7| = 3\sqrt{10}$. Since $a_{26} = a_{12} * r^{14}$, it follows that $|a_{26}| = |a_{12}| * |r^{14}| = (30)(90) = 2700$.

5.) If $\cos(\theta) + 3\sin(\theta) = 2$, then the sum of all possible values of $\tan(\theta)$ is a and the product of all possible values of $\tan(\theta)$ is b. Find $a - 12b$.

Dividing every term by $\cos(\theta)$ produces $1 + 3\tan(\theta) = 2\sec(\theta)$. Squaring both sides yields $1 + 6\tan(\theta) + 9\tan^2(\theta) = 4\sec^2(\theta)$, which can be written in terms of $\tan(\theta)$ as $1 + 6\tan(\theta) + 9\tan^2(\theta) = 4(\tan^2(\theta) + 1)$, or $5\tan^2(\theta) + 6\tan(\theta) - 3 = 0$.This means that $a = -\frac{6}{5}$ and $b = -\frac{3}{5}$, so $-\frac{6}{5} - 12\left(-\frac{3}{5}\right) = 6$.

6.) One infinite geometric series has a first term of 7. A second infinite geometric series has a first term of 4. The ratios of the series are opposite numbers, and the infinite sum of the second series is one less than that of the first series. If the first series has a sum that can be written in simplest form as $\frac{a+\sqrt{b}}{c}$ where a, b, and c are integers and b has no perfect square factors greater than 1, find the value of $a + b + c$.

Let the ratio of terms for the first series be r. From the problem description we get $\frac{7}{1-r} = \frac{4}{1+r} + 1$, which after multiplying each term by $1 - r^2$ becomes $7(1 + r) = 4(1 - r) + 1 - r^2$ or $r^2 + 11r + 2 = 0$. This gives values of $r = \frac{-11 \pm \sqrt{113}}{2}$. Note that for the series to converge, $-1 < r < 1$ so $r = \frac{-11+\sqrt{113}}{2}$. This makes the sum of the first series $\frac{7}{1-\left(\frac{-11+\sqrt{113}}{2}\right)}$, which simplifies to $\frac{14}{13-\sqrt{113}} = \frac{13+\sqrt{113}}{4}$, so $a + b + c = 13 + 113 + 4 = 130$.