FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 4 Round 1
Arithmetic: Basic Statistics

1.) $\{104,102,100\}$

2.) $\{166,266,366\}$

3.) $\{1365,745,240\}$

Note: Solutions provided are for Form A only. All forms have similar solution methods.

1.) Consider the following sequence of numbers: $15, 21, 9, 8, m$, and n. If this sequence has a range of $\{30,28,26\}$ and a median of 14, find the maximum value of the sum of the numbers.

For the sum to be maximized, 8 should be the lowest number, making the highest number 38. Since there are six numbers, the median will be the arithmetic mean of the third and fourth largest, which will be one unknown number and 15, making the unknown number 13. Therefore the largest possible sum is $8 + 9 + 13 + 15 + 21 + 38 = 104$.

2.) A sequence of consecutive positive integers with a minimum value of a and a maximum value of b has a range equal to its arithmetic mean. If $b \leq \{500,800,1100\}$, find the largest possible value of a.

Because the integers are consecutive, the arithmetic mean will also be the arithmetic mean of the minimum and maximum values. Therefore, $b - a = \frac{a+b}{2}$, or $b = 3a$. Therefore, for a and b to both be integers, b must be divisible by 3. If $b \leq 500$, then the largest value of a will be one-third the largest possible value of b, which is 498, making $a = 166$.

3.) The geometric mean of a set of n numbers is defined as the nth root of the product of the n numbers. A geometric sequence of 9 numbers has a smallest number of {4,6,9} and a geometric mean of $\{196,150,81\}$. The second largest of the numbers can be written in simplest radical form as $a\sqrt{b}$ with integers a and b. Find $a-b$.

One can multiply the numbers $(A)(Ar)\ldots(Ar^8)$ and then take the ninth root of the product to find that the geometric mean is Ar^4, or one could note that if the sequence is geometric, the geometric mean of the sequence will be the geometric mean of the minimum and maximum values. Since $A=4$, we know $196=4r^4$, making $r=\sqrt{7}$. Therefore the second largest value will be $4\left(\sqrt{7}\right)^7=4*7^3\sqrt{7}=1372\sqrt{7}$, so $a-b=1365$.

FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 4 Round 2 Algebra 1: Quadratic Equations

1.) $\{100,125,150\}$

2.) $\{113,85,61\}$

3.) $\{55,50,105\}$

Note: Solutions provided are for Form A only. All forms have similar solution methods.

1.) If a number x is increased by $(4x)\%$, the result is equivalent to $\{2,3,4\}$ more than twice x. This relationship can be represented by the quadratic equation $x^2 + Ax + B = 0$, where A and B are integers. Find $|2A + B|$.

By the problem description, $\left(1 + \frac{4x}{100}\right)x = 2 + 2x$, or $x + \frac{1}{25}x^2 = 2 + 2x$. Collecting like terms on one side and multiplying every term by 25 yields $x^2 - 25x - 50 = 0$, so $|2A + B| = |2(-25) - 50| = 100$.

2.) The quadratic equation $x^2 - \{16,14,12\}x + p = 0$ has a solution for x of the form $a + bi$ where $a, b,$ and p are positive real numbers. If $b = a - 1$, find the value of p.

Because our two solutions are of the form $a \pm bi$, their sum is $2a$, so $2a = 16$ and therefore $a = 8$. That makes the solutions $8 \pm 7i$. The constant p is the product of the solutions, so $p = 8^2 + 7^2 = 113$.

3.) The quadratic equation $\{(a+1)x^2 - 3\sqrt{b}x + 1 = 0, (a-1)x^2 - 3\sqrt{b}x + 1 = 0, (2a+1)x^2 - 3\sqrt{b}x + 1 = 0\}$ has only one real solution for x. If a and b are positive integers less than 100, then the smallest possible value for x can be written as $\frac{\sqrt{m}}{n}$ where m and n are integers and m has no perfect square factors greater than 1. Find $2n - m$.

The discriminant of the quadratic must be zero, so $\left(3\sqrt{b}\right)^2 - 4(a+1) = 0$, or $4a - 9b = -4$. By inspection we see that $a = 8$ and $b = 4$ satisfies the equation (which would have a solution of $x = \frac{1}{3}$). We can find larger pairs by incrementing a by 9 and b by 4, and the largest valid pair where each number is less than 100 is $a = 98$ and $b = 44$. This gives the quadratic $99x^2 - 6\sqrt{11}x + 1 = 0$, which has a solution $x = \frac{1}{3\sqrt{11}} = \frac{\sqrt{11}}{33}$, so $2(33) - 11 = 55$.

FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 4 Round 3 Geometry: Similarity

1.) $\{80{,}320{,}500\}$

2.) $\{23{,}11{,}19\}$

3.) $\{12{,}20{,}15\}$

Note: Solutions provided are for Form A only. All forms have similar solution methods.

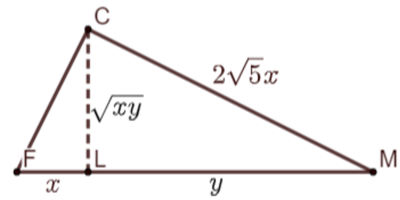
1.) A right triangle has a hypotenuse of length 30 and an area of 180. A similar right triangle has a hypotenuse of $\{20{,}40{,}50\}$. What is its area?

Since the hypotenuse of the second triangle is $\frac{2}{3}$ the length of the hypotenuse of the first, the area of the second triangle will be $\frac{4}{9}$ the area of the first, and $\left(\frac{4}{9}\right)180 = 80$.

2.) A conical tank of height $\{24{,}15{,}21\}$ feet is filled to the top with water. If one-ninth of the water is drained out, then the height of the water is $a\sqrt[3]{b}$ feet where a and b are integers and b has no perfect cube factors larger than 1. Find $2a - 3b$.

After draining, the resulting similar cone of water will have $\frac{8}{9}$ the volume it once did, so the height will be $\frac{2}{3^{2/3}}$ the original height of 24 feet. This ratio can be rationalized to $\frac{2\sqrt[3]{3}}{3}$, and setting equal to $\frac{h}{24}$ gives a height of $16\sqrt[3]{3}$, so $2(16) - 3(3) = 23$.

3.) Consider right triangle FCM with right angle C. The altitude from point C intersects the hypotenuse at point L. If $CM = \{2\sqrt{5}, \sqrt{6}, 2\sqrt{3}\}FL$ and the area of triangle FCM is $\{180, 300\sqrt{2}, 150\sqrt{3}\}$, find CL.



See the diagram (not drawn to scale). We will label FL as x, making $CM = 2\sqrt{5}x$. We will then label LM as y, making $CL = \sqrt{xy}$ by similarity. We then have $y^2 + xy = 20x^2$, which factors into $(y + 5x)(y - 4x) = 0$, so $y = 4x$. Therefore $FM = 5x$ and $CL = 2x$. Using the area, we have $\frac{1}{2}(5x)(2x) = 180$, making $x = 6$, and therefore $CL = 12$.

FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 4 Round 4
Algebra 2: Variation

1.) $\{144,16,9\}$

2.) $\{21,12,23\}$

3.) $\{65,75,85\}$

Note: Solutions are for Form A only. All forms have similar solution methods.

1.) If y varies inversely as the square of x and $y = 36$ when $x = 8$, find the value of y when $x = \{4,12,16\}$.

Since $(36)(8^2) = y(4^2)$, it follows that $y = 144$.

2.) The kinetic energy of a moving object varies jointly as its mass and as the square of its speed. If the mass of an object were to increase by $\{60,30,70\}$ percent while its kinetic energy remained constant, by what percent would its speed decrease? Round your answer to the nearest whole percent.

Defining m as mass and s as speed, we have $ms^2 = (1.6m)((1-p)s)^2$. This means $(1-p)^2 = \frac{1}{1.6}$, making $p = .2094$, which when written as a percent and rounded to the nearest whole number makes 21.

3.) Researchers studying the eastern featherhead bird notice that starting at the age of 2 years, the length of the head feather in centimeters varies directly as the square root of the age in years, and the weight of the bird in ounces varies directly as the cube of the length of the head feather in centimeters. If a four-year-old bird has a weight in ounces that is $\{26,30,34\}$ times the length of the head feather in centimeters, find the ratio of the weight in ounces to the head feather length in centimeters of a 10-year-old bird.

Let x be the age of the bird in years, y be the length of the head feather in centimeters, and z be the weight in ounces. We have $y = k_1\sqrt{x}$ and $z = k_2y^3 = k_2k_1^3x^{3/2}$. The ratio of weight to head feather length is then $k_2k_1^2x$, meaning this ratio varies directly as age. Since the ratio is 26 when $x = 4$, then when $x = 10$, the ratio is $\frac{10}{4}(26) = 65$.

FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 4 Round 5 Precalculus: Trig Expressions & De Moivre's Theorem

1.) $\{80,112,42\}$

2.) $\{25,73,49\}$

3.) $\{41,78,125\}$

Note: Solutions provided are for Form A only. All solutions have similar solution methods.

1.) If A is an angle such that $\sin(A) = \left\{\frac{3}{7}, \frac{5}{8}, \frac{2}{5}\right\}$, then $\cos(2A) = \frac{p}{q}$ where p and q are integers with no common factors greater than 1. Find $p + q$.

Since $\cos(2A) = 1 - 2\sin^2(A)$, $\cos(2A) = 1 - 2\left(\frac{3}{7}\right)^2 = 1 - \frac{18}{49} = \frac{31}{49}$, so $p + q = 31 + 49 = 80$.

2.) One of the complex eighth roots of $\sqrt{3} - i$ lies closest to the {negative y-,positive y-,negative x-}axis compared to the other complex eighth roots. The argument for this number is $\frac{a\pi}{b}$, where a and b are integers with no common factors greater than 1. Find $2b - a$.

Since $\sqrt{3} - i$ has an argument of $\frac{11\pi}{6}$, the complex eighth roots will have arguments of $\frac{11\pi}{48} + \frac{2k\pi}{8}$, the arguments have values $\frac{11\pi}{48}, \frac{23\pi}{48}, \frac{35\pi}{48}, \frac{47\pi}{48}, \frac{59\pi}{48}, \frac{71\pi}{48}, \frac{83\pi}{48}$, and $\frac{95\pi}{48}$. The argument closest to $\frac{3\pi}{2}$ is $\frac{71\pi}{48}$, and $2(48) - 71 = 25$.

3.) If x is an angle in quadrant I such that $\tan\left(x+\frac{\pi}{4}\right) = \tan(x) + \{4,6,8\}$, then $\sec^2(x) = a - b\sqrt{c}$ where a, b, and c are integers and c has no perfect square factors greater than 1. Find $a + 2b + 3c$.

Using the formula for the tangent of the sum of two angles, we get $\frac{\tan(x)+1}{1-\tan(x)} = \tan(x) + 4$. Multiplying both sides by $1 - \tan(x)$ gives $\tan(x) + 1 = 4 - 3\tan(x) - \tan^2(x)$, which can be rewritten as $\tan^2(x) + 4\tan(x) - 3 = 0$. This yields $\tan(x) = \sqrt{7} - 2$ (forgoing the extraneous $-\sqrt{7} - 2$). Then $\sec^2(x) = 1 + \tan^2(x) = 1 + (\sqrt{7} - 2)^2 = 12 - 4\sqrt{7}$, and $12 + 2(4) + 3(7) = 41$.

FAIRFIELD COUNTY MATH LEAGUE 2021-22

Match 4 Round 6
Miscellaneous: Conics

1.) $\{200,128,288\}$

2.) $\{3,13,23\}$

3.) $\{20,8,2\}$

Note: Solutions provided are for Form A only. All forms have similar solution methods

1.) An ellipse with equation $\frac{x^2}{\{100,64,144\}} + \frac{y^2}{b} = 1$ has foci that are 6 units apart. Find the sum of all possible values of b.

Either $b - 100 = 9$ or $100 - b = 9$, making the desired value $109 + 91 = 200$.

2.) Circle P has the equation $x^2 + y^2 - 4x - 6y = 3$. Line l intersects circle P at two points. These points are also shared by Circle Q, which has the same radius as circle P but whose center is shifted {1,2,3} right and {2,3,2} down compared to that of circle P. Line l has equation $Ax + By = C$ where $A > 0$ and $A, B,$ and C are integers share no common factors greater than 1. Find the value of $A - B + C$.

Put in standard form, we get that the equation of Circle P is $(x-2)^2 + (y-3)^2 = 16$, making the center $(2,3)$ and the radius 4. Circle Q will then have center $(3,1)$ and radius 4, making its equation $(x-3)^2 + (y-1)^2 = 16$. Expanding this gives $x^2 + y^2 - 6x - 2y = 6$. Subtracting this equation from the expanded form of the equation for circle P gives $2x - 4y = -3$, which is the equation for line l, so $2 - (-4) + (-3) = 3$.

3.) A particular parabola has a focus at $(3,5)$, an x −intercept of $(\{9,7,6\}, 0)$ and a directrix with equation $x = k$. Find the product of all possible values of k.

Since the distance from (9,0) to the directrix must be equal to the distance from (9,0) to (3,5), it follows that $(9-k)^2 = (9-3)^2 + (0-5)^2$, so $(9-k)^2 = 61$. Expanding the binomial square gives $k^2 - 18k + 20 = 0$, making the product of the two possible values of k equal to 20.

Team Round

FAIRFIELD COUNTY MATH LEAGUE 2021-2022 Match 4 Team Round

1.) 45	4.) 98
2.) 55	5.) 20
3.) 287	6.) 16744

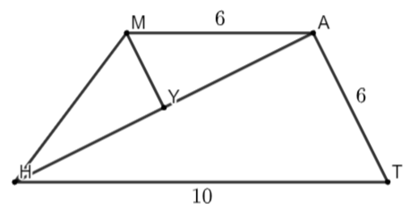
1.) The geometric mean of a set of n numbers is defined as the nth root of the product of the n numbers. A particular sequence of n numbers $\{a_1, \dots, a_n\}$ has the property that $a_m = k^{(-1)^{m+1}m}$ for $m \in \{1, \dots, n\}$ where k is a positive integer. If the sequence has a median of $\frac{730}{162}$ and a geometric mean of p, find the value of $\frac{5}{p^2}$.

The numbers of this sequence will be $\{k, k^{-2}, k^3, \dots\}$. If there is an odd number of terms, the median of this sequence will be k. If there is an even number of terms, the median will be $\frac{k+k^{-2}}{2}$. Since the median is not an integer, we know there is an even number of terms. Therefore $k + k^{-2} = \frac{730}{81}$, or $\frac{k^3+1}{k^2} = \frac{730}{81}$, so $k = 9$. Next we can note that if there is an even number of terms, the geometric mean will always be $k^{-1/2}$, which in this case will be $\frac{1}{3}$, making the desired value $5 * 3^2 = 45$.

2.) The quadratic equation $x^2 + 3x - 5 = 0$. This quadratic has two real zeros. Another quadratic equation $Ax^2 + Bx + C = 0$, where $A, B,$ and C are relatively prime integers, has real zeros that are exactly k times the zeros of the first quadratic where k is an integer greater than 1. If $|A| + |B| + |C| < 1000$, how many values of k are there?

The solutions to the original quadratic have a sum of magnitude 3 and a product of magnitude 5. If the solutions were multiplied by k, the magnitude of the sum of the zeros would be $3k$ and the magnitude of the product would be $5k^2$. The sum of these coefficient magnitudes would be $1 + 3k + 5k^2$. The largest value of k that keeps this sum below 1000 is 13, so $k \in \{2,3,4,5,6,7,8,9,11,12,13\}$. This makes a total of 12 possible values of k.

3.) Consider trapezoid $MATH$ with bases $\overline{MA}$ and $\overline{TH}$. $MA = 6, TH = 10$, and $\overline{MA} \cong \overline{AT}$. If diagonal $\overline{HA}$ creates right angle HAT, then HM can be written in simplest radical form as $\frac{a\sqrt{b}}{c}$. Find $a + 2b - c$.



Refer to the diagram (not to scale). We can draw $\overline{MY}$ such that $\overline{MY} \perp \overline{AH}$. Then triangle MAY is similar to triangle THA. We see that $AH = 8$, and since $\frac{MA}{AY} = \frac{TH}{HA}$, we get $AY = 4.8$, which means $HY = 3.2$. We can solve for MY using similarity or the Pythagorean theorem to get $MY = 3.6$. Therefore $HM = \sqrt{3.2^2 + 3.6^2} = \frac{2\sqrt{145}}{5}$, so $2 + 2(145) - 5 = 287$.

4.) *Best Boxes, Inc.* measures boxes (rectangular prisms) by their "boxiness", which they say varies directly as the square of the length and the square root of their width and inversely as the height. Obviously the boxiness of a box changes with its orientation. A particular box with a volume of 10, three dimensions of different sizes, and one dimension of size 2 has an interesting property: it has the same boxiness when its length is 2 as it does when its width is 2. The length of the box's longest side can be expressed as $\frac{\sqrt[3]{a}}{b}$ where a and b are positive integers and a has no perfect cube factors greater than 1. Find $a - b$.

The "boxiness" of a box varies as the expression $\frac{l^2\sqrt{w}}{h}$ where l is the length, w is the width and h is the height. We have $\frac{2^2\sqrt{a}}{b} = \frac{b^2\sqrt{2}}{a}$ (we know that a and 2 did not simply switch because then $a = 2$ and the box would not have three dimensions of different sizes). Since $b = \frac{5}{a}$, we have $\frac{4\sqrt{a}}{\frac{5}{a}} = \frac{\frac{25}{a^2}\sqrt{2}}{a}$, which gives $a^{9/2} = \frac{125\sqrt{2}}{4} = \frac{5^3}{2^{3/2}}$, so $a = \frac{5^{2/3}}{2^{1/3}} = \frac{\sqrt[3]{100}}{2}$. This is the largest of the three dimensions, since the other two values are 2 and $\sqrt[3]{10}$. Therefore $a - b = 100 - 2 = 98$.

5.) How many complex numbers z have the property that $z^{100} = 1$ and $z^{2021} = z$?

From the first equation we know that z has a modulus of 1 and must have an argument of $\frac{2k\pi}{100}$ where $k = 0, \ldots, 99$. From the second equation we get $z^{2020} = 1$, and since $z^{2020} = (z^{2000})(z^{20}) = (z^{100})^{20}(z^{20}) = z^{20} = 1$, it also follows that z must have an argument of the form $\frac{2k\pi}{20}$ for $k = 0, \ldots, 19$. Since each of these arguments is also in the first set, all 20 of them are valid, giving 20 total complex numbers that satisfy both statements.

6.) Two strobe lights that are blinking simultaneously are 275 miles apart. A ship is navigating along a path so that there is a constant difference of $.0005$ seconds between when it receives a signal from either light. If the lights are placed on the x- axis then the path of the ship can be modeled by the equation $\frac{x^2}{A} - \frac{y^2}{B} = 1$. What is the value of B? (Use the fact that the speed of light is $186{,}000$ miles per second. Assume the curvature of the earth is negligible.)

Note that the coordinates of the foci (the lights) would be $(\pm 137.5, 0)$. Multiplying $(186{,}000)(.0005)$ gives us a distance of 93 miles, which is the difference in the ship's distance from either light along its path. If the ship were on the x −axis and its distance from the closer light is k, the distance from the other light is $275 - k$. We have it that $(275 - k) - k = 93$ so $k = 91$. Therefore the distance from $(0,0)$ to where the ship would be on the x −axis is $\frac{275}{2} - 91 = 46.5$. Therefore we have $\frac{x^2}{46.5^2} - \frac{y^2}{B} = 1$. The value of B is found using $46.5^2 + B = 137.5^2$, so $B = 16744$.