FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 3 Round 1
Arithmetic: Scientific and base notation

1.) {16,20,12}

2.) {111,112,120}

3.) {9,7,8}

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) If the product $2^{\{13,16,10\}} * 49^{2y} * 5^{3x}$ can be written in scientific notation as $1.4 * 10^n$ for some integer value n, find the value of $\frac{x}{y}$.

Simplifying the relationship to prime bases gives $2^{13} * 7^{4y} * 5^{3x} = 2 * 7 * 2^{-1} * 5^{-1} * 2^n * 5^n$, or $2^{13} * 5^{3x} * 7^{4y} = 2^n * 5^{n-1} * 7$. From this we know $y = \frac{1}{4}$, $n = 13$, and $3x = 12$ so $x = 4$. Therefore $\frac{x}{y} = 16$.

2.) If n is a positive integer base such that $\{10_n(114_n) = (1000_2)(10_n)(15_n) - 3330_4, 10_n(110_n) = 1001_2(10_n)(14_n) - 10200_4, 10_n(114_n) = 1010_2(10_n)(14_n) - 11010_4\}$, find the sum of all possible values of n. Enter your result as a numeral in base 3.

Writing the equation in base 10 as an algebraic relationship with n gives the equation $(n)(n^2 + n + 4) = 8(n)(n + 5) - 252$, which upon expanding and setting equal to zero gives $n^3 - 7n^2 - 36n + 252 = 0$, which factors by grouping into $(n^2 - 36)(n - 7) = 0$, so $n = 6, -6$, or 7. Since -6 is extraneous, the sum of possible values is 13, or in base 3, $9 + 3 + 1$ or 111.

3.) Consider the repeating decimal $.\overline{aa0}_n$, where n is a positive integer base and a is a digit such that $a = n - 1$. This decimal can be written as a fraction in base 10 as $\frac{p}{q}$ where p and q are numerals in base 10 with no common factors greater than 1. If $p + q = \{181,113,145\}$, find the value of n.

Since $.\overline{aa0}_n = .aa_n \sum_{k=0}^{\infty} \left(\frac{1}{n^3}\right)^k = \left(\frac{n-1}{n} + \frac{n-1}{n^2}\right) \sum_{k=0}^{\infty} \left(\frac{1}{n^3}\right)^k$ this repeating decimal can be expressed in base 10 as a function of n with the expression $\frac{\frac{n-1}{n}+\frac{n-1}{n^2}}{1-\frac{1}{n^3}}$. Multiplying both the numerator and denominator by n^3 and simplifying produces $\frac{n^3-n}{n^3-1}$, or $\frac{n(n-1)(n+1)}{(n-1)(n^2+n+1)} = \frac{n^2+n}{n^2+n+1}$. Therefore $q = p + 1$, so $p = 90$ and $q = 91$. Since $n^2 + n = 90$, it follows that $n = 9$.

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Match 3 Round 2 Algebra 1: Word Problems

1.) $\{4,5,3\}$

2.) $\{80,60,40\}$

3.) $\{60,40,36\}$

Note: Solutions provided are for Form A only. All forms have similar solution methods.

1.) A boat that would be traveling 20 mph in still water takes $\{5,8,12\}$ minutes longer to travel against a river's current from Pier A to Pier B than to travel from Pier B to Pier A. If the current has a constant speed of $\{4,5,10\}$ miles per hour, how far in miles is it from Pier A to Pier B?

Let the time in hours to travel from Pier B to Pier A (with the current) be t. Then $24t = 16(t + \frac{1}{12})$, or $8t = \frac{4}{3}$, or $t = \frac{1}{6}$. Therefore the distance between the piers is $24\left(\frac{1}{6}\right) = 4$ miles.

2.) Mr. Hill's property contains an equal number of oak trees and maple trees, but the oak leaves take longer to rake. After the maple trees lose half their leaves and the oak trees lose one-third of their leaves, it takes Mr. Hill 50 minutes to rake them all. After all of the remaining maple and oak leaves have fallen, it takes Mr. Hill an additional $\{80,85,90\}$ minutes to rake them. How many minutes would it have taken Mr. Hill to rake the leaves of all the trees if all of the oak trees were maple trees?

Let the time in minutes to rake the leaves of all the maple trees be m and let the time in minutes to rake the leaves of all the oak trees be k. Then we have that $\frac{1}{2}m + \frac{1}{3}k = 50$ and $\frac{1}{2}m + \frac{2}{3}k = 80$. Subtracting the e4quations yields $\frac{1}{3}k =$

30, which makes $\frac{1}{2}m = 20$. Since the problem asks for the time required if all the oak trees were maple trees, this is equivalent for asking for the value of $2m$, which is $4(20) = 80$.

3.) A particular field has grass that grows at a continuous constant rate. If the field starts with {12,30,60} days of grass growth, then it would take 30 cows feeding continuously to clear it in 20 days, or 40 sheep eating continuously to clear it in 36 days. How many days would it take for 10 cows and 20 sheep to clear it with the same amount of starting growth?

Let g represent the rate of grass growth per day, c represent the rate at which one cow eats grass per day, and s represent the rate at which one sheep eats grass day. We know 30 cows would eat the equivalent of $12 + 20 = 32$ days of grass growth in 20 days, so $30(20)c = 32g$. We also know that 40 sheep would eat the equivalent of $12 + 36$ days of grass growth in 36 days, so $40(36)s = 48g$. Letting n be the required number of days, we get $(12 + n)g = n(10c + 20s)$. Substituting $10c = \frac{32}{60}g$ and $20s = \frac{48}{72}g$ gives $(12 + n)g = ng\left(\frac{8}{15} + \frac{2}{3}\right)$, or $12 + n = \frac{6}{5}n$, which gives $n = 60$.

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Match 3 Round 3
Geometry: Polygons

1.) $\{144,150,156\}$

2.) $\{320,336,288\}$

3.) $\{26,28,30\}$

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) If the measure of one exterior angle of a regular n-gon in degrees is exactly $\{3.6,2.5,1.6\}n$, find the measure of one interior angle of the n-gon in degrees.

Setting up the relationship $\frac{360}{n} = 3.6n$, we get $360 = 3.6n^2$, so $n^2 = 100$ and therefore $n = 10$.

2.) A particular $\{38,65,20\}$-gon has the property that the measures of 5 of its angles in degrees form an arithmetic sequence, with the largest angle having twice the measure of the smallest. The remaining angles are congruent to the smallest angle in the arithmetic sequence. Find the measure in degrees of the largest angle of the polygon.

Let a be the measure of the smallest angles of the polygon. Then the five measures making up the arithmetic sequence will be $a, 1.25a, 1.5a, 1.75a,$ and $2a$. This sequence has a sum of $7.5a$. Therefore $(33 + 7.5)a = 36(180)$, giving $a = 160$, so the measure of the largest angle in degrees is 320.

3.) A particular regular n-gon has interior angles whose difference in measure in degrees from that of one of its exterior angles is k $(k > 3)$ times that of the measure of one of its exterior angles. If the number of diagonals of the n-gon is $\{35,26,15\}$ more than six times the number of diagonals of a k-gon, find the value of n.

From the first relationship, we get $180 - \frac{360}{n} - \frac{360}{n} = k\left(\frac{360}{n}\right)$. Dividing every term by 180 and rearranging gives $n = 2k + 4$. From the second relationship we get $\frac{n(n-3)}{2} = 35 + 6 * \frac{k(k-3)}{2}$. Substituting for n and simplifying the fractions gives $(k+2)(2k+1) = 35 + 3k(k-3)$, and expanding and setting equal to zero gives $k^2 - 14k + 33 = 0$. This factors into $(k-3)(k-11) = 0$, giving $k = 3$ or $k = 11$, but the restriction on k means $k = 11$, so $n = 26$.

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Match 3 Round 4 Algebra 2: Functions and Inverses

1.) $\{21,35,7\}$

2.) $\{3,2,1\}$

3.) $\{18,25,21\}$

Note: The solutions provided are for Form A only. All forms have similar solution methods.

Note: The inverse f^{-1} of a function $f(x)$ is not necessarily a function.

1.) Let $f(x) = \frac{4}{3}x - \left\{\frac{3}{2}, \frac{5}{2}, \frac{1}{2}\right\}$. If (a, b) is an ordered pair on both $f(x)$ and $f^{-1}(x)$, find $2f(2a)$.

The only way that (a, b) can be on a linear function and on its inverse is if $a = b$, so $a = \frac{4}{3}a - \frac{3}{2}$, so $a = \frac{9}{2}$. Therefore $2f(9) = 2\left(\frac{21}{2}\right) = 21$.

2.) The function $f(x) = \frac{1}{\sqrt{x-p}-q}$ has a domain of $[3, \{39,28,19\}) \cup (\{39,28,19\}, \infty)$. What is the value of $q - p$?

The first thing to note is that the domain has a greatest lower bound of 3, which should come from the restriction imposed by the radical expression. This implies that $p = 3$. From there we see that $x = 39$ is excluded from the domain, which comes from the expression in the denominator equaling 0, so $\sqrt{39-3} - q = 0$ gives $q = 6$, and $6 - 3 = 3$.

3.) Consider the functions $f(x) = \frac{3}{x^2+2x+a}$ and $g(x) = \frac{5}{x^2+ax+b}$. If $f(x)$ has a range of $\left(0, \left\{\frac{3}{2}, \frac{3}{4}, \frac{3}{8}\right\}\right]$ and $g(x)$ has a range of $\left(-\infty, \left\{-\frac{4}{5}, -\frac{4}{9}, -\frac{4}{17}\right\}\right] \cup (0, \infty)$, find the value of $2a - 3b$.

Starting with $f(x)$, we notice that the quadratic in the denominator must be positive for all x since the range of $f(x)$ is strictly positive. In addition, the quadratic must have a minimum value of 2 since this gives the maximum value of $f(x)$ of $\frac{3}{2}$. Noticing that that minimum value of the quadratic occurs at the vertex when $x = -1$, we get $(-1)^2 + 2(-1) + a = 2$, so $a = 3$. Reasoning for $g(x)$ is similar, and now the minimum value of the quadratic will be $-\frac{25}{4}$ to produce the least upper bound of the lower part of the range. The minimum value of the quadratic will occur when $x = -\frac{3}{2}$, and since $\left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) + b = -\frac{25}{4}$, we get $b = -4$. Therefore $2(3) - 3(-4) = 18$.

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Match 3 Round 5 Precalculus: Exponents & Logarithms

1.) $\{57,75,63\}$

2.) $\{144,400,784\}$

3.) $\{75,108,48\}$

Note: Solutions provide are for Form A only. All forms have similar solution methods.

1.) If $\log_9(6) = \{3x+4, 2x+9, 5x+1\}$, then $\log_3(8) = ax + b$ where a and b are integers. Find $2a + b$.

From the first equation, we have $9^{3x+4} = 6$, which means $3^{6x+8} = 2 * 3$, or $3^{6x+7} = 2$. Then cubing both sides gives $3^{18x+21} = 8$, so $\log_3(8) = 18x + 21$, giving $2(18) + 21 = 57$.

2.) For $x > \frac{1}{3}$, the function $f(x) = \log_5(3\mathrm{x} - 1) + \log_{25}(2\mathrm{x} + \{1,3,5\})$ can be written as $f(x) = A\log_5(ax^3 + bx^2 + cx + d)$. Find the value of $(a + b + c + d)^{1/A}$.

Note that $\log_{25}(a) = \frac{1}{2}\log_5(a)$, so $f(x) = \log_5(3x - 1) + \frac{1}{2}\log_5(2x + 1)$, so $f(x) = \frac{1}{2}\log_5((3x - 1)^2) + \frac{1}{2}\log_5(2x + 1) = \frac{1}{2}\log_5((3x - 1)^2(2x + 1)$, and after expanding, $f(x) = \frac{1}{2}\log_5(18x^3 - 3x^2 - 4x + 1)$, giving $(18 - 3 - 4 + 1)^2 = 144$.

3.) For a particular positive value of n, the equation $\log_{\mathrm{x}}(\{625,1296,256\}) + \log_{\{5,6,4\}}(x^9) = n$ has only one real solution for x. If this solution is $x = k$, find the value of $3k^3$.

Using properties of logarithms, $\frac{\log_5(625)}{\log_5(x)} + 9\log_5(x) = n$, and evaluating the given logarithm and multiplying everything by $\log_5(x)$ gives $4 + 9(\log_5(x))^2 = n\log_5(x)$. Setting equal to zero gives $9(\log_5(x))^2 - n\log_5(x) + 4 = 0$. If $n = 12$, this becomes $(3\log_5(x) - 2)^2 = 0$ and will have exactly one solution for x, namely $\log_5(x) = \frac{2}{3}$ so $x = 5^{2/3}$. Therefore $3\left(5^{\frac{2}{3}}\right)^3 = 75$.

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Match 3 Round 6
Miscellaneous: Matrices

1.) $\{12,13,14\}$

2.) $\{19,55,33\}$

3.) $\{354,311,209\}$

Note: Solutions provided are for Form A only. All forms have similar solution methods.

1.) If $\begin{bmatrix} m & 2 & 5 \\ -1 & n & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ m \end{bmatrix} = \begin{bmatrix} \{16,4,-8\} \\ 71 \end{bmatrix}$, find the value of n.

From the first row of the first matrix, we have $2m + 12 + 5 + 3 = 16$, so $m = -2$. From the second row, we have $-2 + 6n + 3 - 2 = 71$, so $n = 12$.

2.) If matrix $A = \begin{bmatrix} 3 & -\frac{5}{3} \\ -1 & \frac{2}{3} \end{bmatrix}$, matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $AB = A^{-1}$, find the value of $\{a, b, c\}$.

Noting that $A(A^{-1}) = I$ and therefore $A(A^{-1})(A^{-1}) = I(A^{-1}) = A^{-1}$, it follows that $B = (A^{-1})^2$. Since $Det(A) = \frac{1}{3}$, $A^{-1} = 3\begin{bmatrix} \frac{2}{3} & \frac{5}{3} \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}$. Squaring the matrix makes $\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 19 & 55 \\ 33 & 96 \end{bmatrix}$, so $a = 19$.

3.) The matrix $A = \begin{bmatrix} \{2,3,7\} & b \\ 5 & a \end{bmatrix}$ has all positive integer entries whose sum is less than 1000 and a determinant of 2. Find the maximum value of the sum of the main diagonal entries of A^{-1}.

From the determinant we know that $2a - 5b = 2$, which is a Diophantine equation. By inspection, $a = 6$ and $b = 2$ satisfies the equation, which gives the matrix an entry sum of 15. Since finding the next largest values of a and b requires incrementing a by 5 and b by 2, this would increase the entry sum of 7. We can set up $15 + 7n < 1000$ to find that we can increment the value by at most 140 times to keep the entry sum below 1000. This makes the maximum value of $a = 6 + 140(5) = 706$. Since the determinant of A^{-1} is $\frac{1}{2}$, it follows that the maximum sum of the main diagonal entries of A^{-1} is $\frac{1}{2}(2 + 706) = 354$.

Team Round

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1.) 12	4.) 9
2.) 1125	5.) 64
3.) 135	6.) 14

1.) For particular values x and y, $(6^{2x})(18^{y})(50^{x+y}) = 60^{n} = a * 10^{14}$, where n is a positive integer and $1 \leq a < 10$. Find the value of $x - y$.

Converting the factors of the product to prime bases and simplifying gives $(2^{3x+2y})(3^{2x+2y})(5^{2x+2y})$, and since $60^{n} = (2^{2n})(3^{n})(5^{n})$, we have $3x + 2y = 2(2x + 2y)$, which simplifies to $x = -2y$. Substituting for x gives 60^{-2y}, and by inspection we see $60^{8} \approx 1.68 * 10^{14}$. Therefore $-2y = 8$ so $y = -4$ and $x = 8$, making $x - y = 12$.

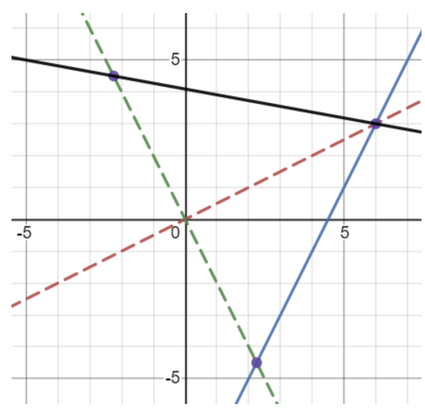
2.) Maddy and Patty go for a run. Maddy runs the first half of her total running distance at an average speed of m miles per hour but then slows down to an average speed of $\frac{1}{2}m$ miles per hour for the remainder of the run. Patty spends the first half her total time? running at an average speed of p miles per hour but then slows down to an average speed of $\frac{1}{2}p$ miles per hour for the remainder of the run. If Maddy and Patty end up running the same total distance in the same total time, find the value of $1000\left(\frac{m}{p}\right)$.

Let D be the total distance each person runs. Using Maddy's information, we find that the total time running can be expressed as $\frac{.5D}{m} + \frac{.5D}{.5m} = \frac{3D}{2m}$. Then using Patty's information, we can set up $D = p\left(\frac{3D}{4m}\right) + .5p\left(\frac{3D}{4m}\right) = p\,\frac{9D}{8m}$, so $1 = \frac{9p}{8m}$, so $\frac{m}{p} = \frac{9}{8}$ and $1000\left(\frac{9}{8}\right) = 1125$.

3.) A regular m-gon and a regular n-gon, where $m \geq n$, have the property that the sum of the measures of one interior angle of each in degrees is equal to the number of diagonals in a 27-gon. If A is the value of m where $m - n$ is maximized, B is the value of m where $m - n$ is minimized, and C is the number of ordered pairs (m, n) that exist, find the value of $A + B + C$.

First we find that the number of diagonals in a 27-gon is $\frac{27*24}{2} = 324$. Then $180 - \frac{360}{m} + 180 - \frac{360}{n} = 324$, which simplifies to $\frac{360}{m} + \frac{360}{n} = 36$. Each term divides by 36 so $\frac{10}{m} + \frac{10}{n} = 1$ and $10m + 10n = mn$. Isolating m gives $m = \frac{10n}{n-10}$. We can use this to find five total ordered pairs (m, n): $(110,11), (60,12), (35,14), (30,15)$, and $(20,20)$. Therefore $A = 110$, $B = 20$, and $C = 5$, so $A + B + C = 135$.

4.) The function $f(x) = 2x - 9$ is reflected across $y = \frac{1}{2}x$ to product the function $g(x)$. Find the value of $g(-27)$.



Refer to the diagram. First note that $f(x)$ intersects the line of reflection at the point (6,3). This point also lies on $g(x)$. We can find another point on $g(x)$ by constructing the line $y = -2x$ and constructing an isosceles triangle with height lying on the line of reflection, making the origin the midpoint of the base of the triangle. Noting that $f(x)$ intersects the line $y = -2x$ at the point $\left(\frac{9}{4}, -\frac{9}{2}\right)$, it follows that $g(x)$ will intersect $y = -2x$ at $\left(-\frac{9}{4}, \frac{9}{2}\right)$. We now have two points from $g(x)$ and can construct the equation: $g(x) = -\frac{2}{11}x + \frac{45}{11}$. Therefore $g(-27) = \frac{54}{11} + \frac{45}{11} = \frac{99}{11} = 9$.

5.) If the function $f(x) = \log_a(2 + 4x - x^2) + \log_a(10 - 4x + x^2)$ has a maximum value of $\frac{1}{3}(\log_2(3) + 1)$, what is the value of a?

Noting $f(x) = \log_a(6 - 4 + 4x - x^2) + \log_a(6 + 4 - 4x + x^2)$, this makes $f(x) = \log_a(6 - (x - 2)^2) + \log_a(6 + (x - 2)^2) = \log_a(36 - (x - 2)^4)$. This function will have a maximum value of $\log_a(36)$ when $x = 2$. We then have $a^{\frac{1}{3}(\log_2(3)+1)} = 6^2$, so $a^{\log_2(6)} = 6^6$. Letting $a = 2^k$, we have $2^{k \log_2 6} = 6^6$, which means $6^k = 6^6$. Therefore $a = 2^6 = 64$.

6.) The nonsingular matrix $A = \begin{bmatrix} k & 3 & 5 \\ 2 & k+1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$ has the property that the determinant is equal to the trace (sum of the main diagonal entries) of the matrix. What is the value of the determinant?

Using the provided relationship, we have $k(k + 1)(4) - 3\big((2)(4) - (-1)(2)\big) + 5\big(-(k + 1)(2)\big) = k + k + 1 + 4$, which simplifies to $4k^2 - 8k - 45 = 0$, which factors into $(2k - 9)(2k + 5) = 0$. This gives possible values of $k = \frac{9}{2}$ or $k = -\frac{5}{2}$. However, $k = -\frac{5}{2}$ makes the trace equal to 0, so matrix A would be singular. Therefore $k = \frac{9}{2}$, and the determinant is $2\left(\frac{9}{2}\right) + 5 = 14$.