FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 2 Round 1
Arithmetic: Factors & Multiples

1.) {72,60,48}

2.) {15360,3600,6480}

3.) {360,264,756}

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) The least common multiple of a and b is {6,5,4} times the greatest common factor of a and b. If $ab = \{864,720,576\}$, find the least common multiple of a and b.

Since $gcf(a, b) * lcm(a, b) = ab$, let $gcf(a, b) = x$. Then $lcm(a, b) = 6x$, and $x(6x) = 6x^2 = 864$, so $x = 12$, and $lcm(a, b) = 6(12) = 72$.

2.) What is the smallest whole number to have exactly {44,45,50} factors, including 1 and itself?

This strategy uses the fact that a number with prime factorization $p_1^{n_1} * p_2^{n_2} * \ldots * p_N^{n_N}$ will have $(n_1 + 1)(n_2 + 1) * \ldots * (n_N + 1)$ factors. The prime factorization of 44 is $2^2 * 11$, so 44 can be obtained using $44 * 1$ or $11 * 4 * 1$ or $11 * 2 * 2$. The smallest whole number with 44 factors using $44 * 1$ is 2^{43}. The smallest whole number with 44 factors using $11 * 4$ is $2^{10} * 3^3 = 27468$. Finally, the smallest whole number with 44 factors using $11 * 2 * 2$ is $2^{10} * 3^1 * 5^1 = 15360$ which is the smallest option.

3.) The greatest common factor of N and $\{54,24,54\}$ is $\{9,4,9\}$. The least common multiple of N and $\{378,120,594\}$ is $\{1890,1320,4158\}$. Find the sum of all possible values of N.

Because the prime factorization of 54 is $2 * 3^3$, we know N has 3^2 in its factorization, but no 2 or third 3. We also know that the prime factorization of 378 is $2 * 3^3 * 7$ and the prime factorization of 1890 is $2 * 3^3 * 5 * 7$. Therefore N MUST have a 5 in its factorization but MAY have a 7. This gives two possible values for N: $3^2 * 5 = 45$ and $3^2 * 5 * 7 = 315$, so the sum is $45 + 315 = 360$.

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Match 2 Round 2
Algebra 1: Polynomials and Factoring

1.) {29,20,53}

2.) {96,160,72}

3.) {38,26,30}

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) If $(2x + 3)(ax + b) = \{8,6,4\}x^2 + cx - \{21,15,27\}$ for all values of x, find the value of $a - 3b - 2c$.

Expanding the left side gives $2ax^2 + (3a + 2b)x + 3b = 8x^2 + cx - 21$. Since $2a = 8$, $a = 4$, and since $3b = -21$, $b = -7$. Therefore $c = 3(4) + 2(-7) = -2$, and $a - 3b - 2c = 4 - 3(-7) - 2(-2) = 29$.

2.) If the polynomial $2x^3 - \{22,26,20\}x^2 + mx - n$ with constant coefficients m and n has three not-necessarily distinct positive integer zeros, what is the largest possible value of n?

We know the sum of the zeros must be $\frac{22}{2} = 11$. Since n will be twice the product of the zeros, we need the three integers (which may repeat) that add to 11 but will give the largest product. This will occur when the numbers are as close in value as possible, which in this case is $4 + 4 + 3$, whose product is $4 * 4 * 3 = 48$. Therefore the largest possible value of n is $2 * 48 = 96$.

3.) A particular quartic polynomial with integer coefficients has a leading coefficient of 1, a cubic coefficient of $\{-12, -16, -14\}$, one zero of $\{2 + i, 3 + i, 2 + 2i\}$, and another zero of $a + bi$ where a and b are nonzero integers and $a \neq \{2,3,2\}$. If the constant term of the quartic is less than 2021, find the number of possible values of b.

Because the leading coefficient of the quartic is 1, then the sum of the zeros must be 12. We know that one zero is $2 + i$, which means another zero must be $2 - i$. These zeros have a sum of 4 and a product of 5. For the sum of all 4 zeros to be 12, the remaining two zeros must have a real coefficient of 4. The product of the zeros will then be $5 * (16 + b^2)$. Setting $5 * (16 + b^2) < 2021$ yields $b^2 < 388.2$. Since $\sqrt{388.2} \approx 19.7$ $-19.7 < b < 19.7$, and since $b \neq 0$, this gives 38 total possibilities.

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Match 2 Round 3
Geometry: Area & Perimeter

1.) $\{36,64,100\}$

2.) $\{177,153,129\}$

3.) $\{112,180,192\}$

Note: Solutions are provided for Form A only. All forms have similar solution methods.

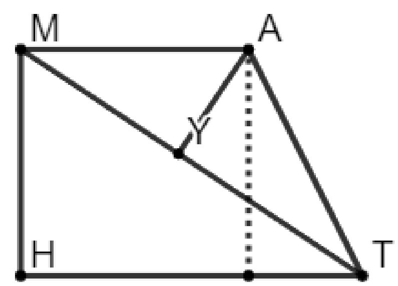
1.) A square with area N has a perimeter equal to the circumference of a circle with diameter $\{6,8,10\}$ and area M. Find the value of $\frac{M^2}{N}$.

The circle in question would have a circumference of 6π and an area of 9π. The square then has a side length of $\frac{3}{2}\pi$ and an area of $\frac{9}{4}\pi^2$. Therefore $\frac{M^2}{N}$ is $\frac{(9\pi)^2}{\frac{9}{4}\pi^2} = 36$.

2.) A regular hexagon has the property that the difference between the longest diagonal length and the perpendicular distance between any two opposite sides is exactly $\{7,6,5\}$ units. The perimeter of the hexagon can be written as $a + b\sqrt{c}$ where a, b, and c are positive integers and c has no perfect square factors greater than 1. Find $a + 2b + 3c$.

Let s the length of one side of the hexagon. The longest diagonal then has length $2s$ and the perpendicular distance from any two opposite sides is $2\left(\frac{\sqrt{3}}{2}s\right) = \sqrt{3}s$, so the difference is $s(2 - \sqrt{3})$. Setting this equal to 7, we see $s = 7(2 + \sqrt{3}) = 14 + 7\sqrt{3}$. Therefore the perimeter is $6s$, or $84 + 42\sqrt{3}$, so $a + 2b + 3c = 84 + 2(42) + 3(3) = 177$.

3.) Consider trapezoid $MATH$ with $\overline{MA} || \overline{HT}$, right angle H, $MA < TH, MA = \{16,18,16\}$, $MH = \{7\sqrt{7}, 12\sqrt{5}, 24\sqrt{3}\}$ and $AT = \{4\sqrt{23}, 6\sqrt{21}, 16\sqrt{7}\}$. Point Y lies on diagonal $\overline{MT}$ such that $\overline{AY} \perp \overline{MT}$. Find $(AY)^2$.



Refer to the diagram. We can first find HT by using the Pythagorean Theorem. Since $x^2 + (7\sqrt{7})^2 = (4\sqrt{23})^2$, we get $x^2 = 25$, so $x = 5$ and $HT = 21$. The area of triangle MAT is $.5(7\sqrt{7})(16) = 56\sqrt{7}$. We can find MT also using $21^2 + (7\sqrt{7})^2 = (MT)^2$, so $MT = 28$. Since the area of triangle MAT can also be found using $.5(MT)(AY)$, it follows that $56\sqrt{7} = .5(28)(AY)$, so $AY = 4\sqrt{7}$, and $(AY)^2 = 112$.

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Match 2 Round 4
Algebra 2: Absolute Value & Inequalities

1.) {91,89,87}

2.) {7,5,9}

3.) {9,6,5}

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) How many integers satisfy the inequality $3|x-11| < \{136,133,130\}$?

Since $|x-11| < \frac{136}{3} = 45.\overline{3}$, the solution set contains 11 as well as 45 integers both above and below 11, so there are 91 total.

2.) The inequality $|x+a| < b$ has a solution set for x of $(a-2, -5a-\{3,1,5\})$. Find the value of b.

First note that $-a$ must be the midpoint of the solution set, so $a-2+(-5a-3) = -2a$, yielding $a = -\frac{5}{2}$. This means that the inequality $\left|x-\frac{5}{2}\right| < b$ has a solution set of $\left(-\frac{9}{2}, \frac{19}{2}\right)$. Since the solution set has a interval width of 14, that means that it spans 7 units on either side from the center, so $b = 7$.

3.) Consider three positive numbers $a, b,$ and c such that $a < b < c$. The minimum value for $x \in [a,c]$ of $f(x) = |x-a| + |x-b| + |x-c|$ is $\{17,20,25\}$. The maximum value of $f(x)$ for $x \in [a,c]$ is $\{30,33,40\}$. Find the value of $|a+c-2b|$.

Refer to the diagram. While we do not know the relative distances between the three points, this solution works regardless. It helps to keep in mind that $f(x)$ is the total distance from any point to all three points. Let us assume that $b - a = u$ and $c - b = v$, and that $v > u$. The minimum value is always when $x = b$, producing a value of $u + v = 17$. The maximum value in this case occurs at $x = c$, giving a value of $u + 2v = 30$. Therefore $v = 13$ and $u = 4$.

So $|a + c - 2b| = |(c - b) - (b - a)| = |13 - 4| = 9$.

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Match 2 Round 5
Precalculus: Law of Sines & Cosines

1.) $\{49,121,1\}$

2.) $\{38249,9961,1321\}$

3.) $\{630, 96, 278\}$

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) Given triangle MRH with $m = 2$, $r = 3$, and $h = 4$, $\sin^2(\{M, R, H\}) = \frac{a}{b}$ where a and b are integers with no common factors greater than 1. Find $b - a$.

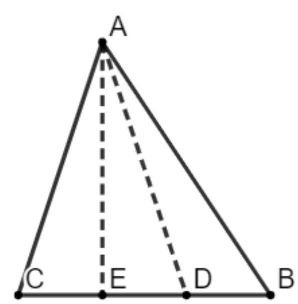
Using the law of cosines, we get $m^2 = r^2 + h^2 - 2rh\cos(M)$, which gives $\cos(M) = \frac{7}{8}$. Then $\sin^2(M) = 1 - \cos^2(M) = 1 - \frac{49}{64} = \frac{15}{64}$, so $b - a = 49$.

2.) Consider triangle ABC with area $\{42\sqrt{5}, 10\sqrt{21}, 6\sqrt{5}\}$. If $\sin(C) = \left\{\frac{3\sqrt{5}}{7}, \frac{\sqrt{21}}{5}, \frac{\sqrt{5}}{3}\right\}$ and both a and b are integers, find the positive difference between the maximum possible value of c^2 and the minimum possible value of c^2.

We can solve for the product ab since $42\sqrt{5} = \frac{1}{2}ab\sin(C) = \frac{1}{2}\left(\frac{3\sqrt{5}}{7}\right)ab$, which gives $ab = 196$. We know that $\cos(C) = \pm\sqrt{1 - \sin^2(C)} = \pm\frac{2}{7}$. The largest possible value of c^2 will occur when a and b are furthest apart, namely 1 and 196, and the angle between them is obtuse, so $\cos(C) = -\frac{2}{7}$. The smallest value of c^2 will occur when a and b are closest in value, namely 14 and 14, and the angle between them is acute, so $\cos(C) = \frac{2}{7}$. In the former case, $c^2 = 1^2 + 196^2 - 2(1)(196)\left(-\frac{2}{7}\right) = 38529$, while in the latter case $c^2 = 14^2 + 14^2 - 2(14)(14)\left(\frac{2}{7}\right) = 280$, and those values have a difference of 38249.

3.) Consider triangle ABC with acute angle B and distinct points D and E on $\overline{BC}$ such that $BD = DE = EC$. If $AB = \{10,15,20\}$, the area of triangle ABC is $\{36,54,72\}$, and $\sin(B) = \frac{24}{25}$, then $\sin(\angle AEB) = \frac{a\sqrt{b}}{c}$ where $a, b,$ and c are integers with a and b having no common factors greater than 1 and b having no perfect square factors other than 1. Find $a + b + c$.

Refer to the diagram (not drawn to scale). We can solve for BC using $36 = \frac{1}{2}(AB)(BC)\sin(B) = \frac{1}{2}(10)\left(\frac{24}{25}\right)(BC)$ to get $BC = \frac{15}{2}$, so $BD = DE = EC = \frac{5}{2}$. From here you can take multiple approaches. One approach is to use the law of cosines to solve for AE, so $AE = \sqrt{10^2 + 5^2 - 2(10)(5)\left(\frac{7}{25}\right)} = \sqrt{97}$, and then use the law of sines since $\frac{\sin(\angle AEB)}{10} = \frac{\frac{24}{25}}{\sqrt{97}}$, so $\sin(\angle AEB) = \frac{240}{25\sqrt{97}}$ or in simplest form $\frac{48\sqrt{97}}{485}$, so $a + b + c = 48 + 97 + 485 = 630$.



FAIRFIELD COUNTY MATH LEAGUE 2021-22

Match 2 Round 6 Miscellaneous: Equations of Lines

1.) {15,12,21}

2.) {56,46,54}

3.) {12,11,7}

Note: Solutions are provided for Form A only. All forms have similar solution methods.

1.) A line perpendicular to $3x - Ay = 24$ but with the same x −intercept has equation $Bx + Cy = \{40,32,56\}$, where $A, B,$ and C are positive numbers. Find the value of AC.

We can see that the x −intercept of the first line is (8,0), so we know $B = 5$. Since the lines are perpendicular, $\frac{3}{A} = \frac{C}{5}$, so $AC = 15$.

2.) A line with a positive slope can be written parametrically as $x = at + 1$ and $y = 6t + b$. If a and b are integers and the line contains the point $\{(10,3), (8,5), (9,7)\}$, find the sum of the greatest possible values of a and b.

We can use the given point to set up $10 = at + 1$ and $3 = 6t + b$ for some value of t. This also means that $\frac{9}{a} = \frac{3-b}{6}$, or $b = 3 - \frac{54}{a}$. Since a and b are integers, the largest possible value of a that produces an integer value for b is 54, which also produces the largest possible integer value for b of 2, so the sum is 56.

3.) A nonzero number m has the property that if a line has a slope of m, any line perpendicular to it will have a slope exactly {3,4,5} less than m. Line a has slope m^2 and y −intercept $(0,-34)$. Line b is perpendicular to a and has y-intercept $(0,\{50,120,127\})$. Find the x −coordinate where lines a and b intersect.

There are many possible approaches to solve this problem. In this case, we use the first relationship to note that $-\frac{1}{m} = m - 3$, or $3 = m + \frac{1}{m}$. Squaring both sides of this equation yields $9 = m^2 + 2 + \frac{1}{m^2}$, or $m^2 + \frac{1}{m^2} = 7$. Then the next statements provide a system of equations $\begin{cases} y = m^2x - 34 \\ y = -\frac{1}{m^2}x + 50 \end{cases}$. We can solve the system by subtracting the equations to yield $0 = \left(m^2 + \frac{1}{m^2}\right)x - 84$. Substituting from earlier gives $0 = 7x - 84$, or $x = 12$.

Team Round

FAIRFIELD COUNTY MATH LEAGUE 2021-2022 Match 2 Team Round

1.) 200

2.) 400

3.) 80

4.) 15

5.) 169

6.) 39

1.) The greatest common factor of 12 and N is 4. If there are at least 175 positive integers less than or equal to 2021 that are divisible by 12 or N find the largest possible value of N.

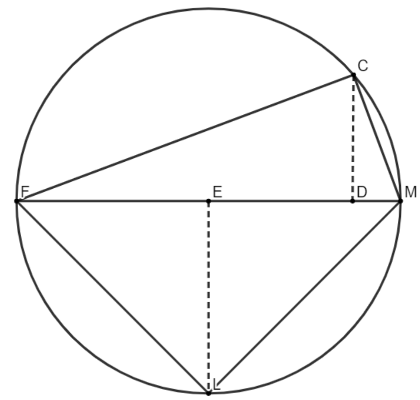
Because $gcf(12, N) = 4$, it follows that $lcm(12, N) = \frac{12N}{4} = 3N$, and N cannot be divisible by 3. Since there are 168 integers less than or equal to 2021 that are divisible by 12 floor $\left(\frac{2021}{12}\right) = 168$, we get that $168 +$ floor $\left(\frac{2021}{N}\right) -$ floor $\left(\frac{2021}{3N}\right) \geq 175$, or floor $\left(\frac{2021}{N}\right) -$ floor $\left(\frac{2021}{3N}\right) \geq 7$. The largest value of N that satisfies this inequality and is not divisible by 3 is 200.

2.) Consider $f(x) = x^4 - 5x^2 + 4$. For how many positive integer values of $n \leq 1000$ is $f(n)$ divisible by 360?

Note that $f(x) = (x-2)(x-1)(x+1)(x+2)$, and is the product of 4 out of 5 consecutive numbers (without the number itself). The prime factorization of 360 is $2^3 * 3^2 * 5$. For $f(n)$ to be divisible by 2^3, n can be odd or an even number but not a multiple of 4, because then $f(n)$ would only be divisible by 2^2. For $f(n)$ to be divisible by 3^2, it is necessary and sufficient that n is not divisible by 3, and for $f(n)$ to be divisible by 5, it is necessary and sufficient that n is not be divisible by 5. Therefore, we need the number of positive integers $n \leq 1000$ such that n is not divisible by 3, 4, or 5. We can find this number by applying the addition rule and subtracting from 1000, giving 400 total possibilities.

3.) Quadrilateral $FCML$ is inscribed in a circle with an area of 50π, and $\overline{FM}$ is a diameter of the circle. The altitude of triangle FCM from C intersects $\overline{FM}$ at D, and the altitude of triangle FLM from point L intersects $\overline{FM}$ at E. If $ED = 4DM$ and $FE = 5DM$, find the area of $FCML$.

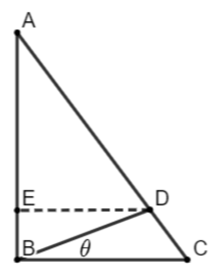
Refer to the diagram (not drawn to scale). We know the radius of the circle is $5\sqrt{2}$, and so $DM = \sqrt{2}$, $ED = 4\sqrt{2}$, and $FE = 5\sqrt{2}$. From here we can take several approaches, but one approach is to use the fact that $(FD)(DM) = (CD)^2 = 18$, so $CD = 3\sqrt{2}$, and the area of triangle FCM is $.5(FM)(CD) = 30$. We can do the same with LE or note that triangle FML is isosceles. Either way the area of triangle FML is $.5(10)(10) = 50$, so the area of $FCML$ is 80.



4.) If, for constants a and b, the solution set for $|x - ab| > b$ is $\left(-\infty, -\frac{2}{3}a\right) \cup \left(\frac{3}{2}b, \infty\right)$, find the value of $10a + 15b$.

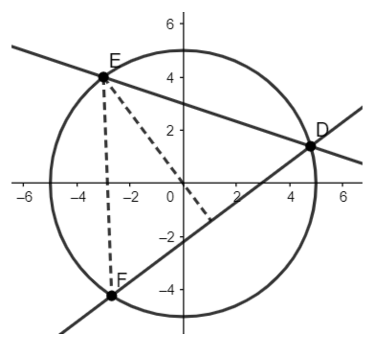
First we note that ab is the midpoint of $-\frac{2}{3}a$ and $\frac{3}{2}b$, so $\frac{3}{2}b - \frac{2}{3}a = 2ab$. We also know that $\frac{3}{2}b - ab = b$, which gives $\frac{1}{2}b = ab$, or $a = \frac{1}{2}$. Substituting this value for a in the first equation gives $\frac{3}{2}b - \frac{1}{3} = b$, or $b = \frac{2}{3}$. Therefore, $10a + 15b = 10\left(\frac{1}{2}\right) + 15\left(\frac{2}{3}\right) = 5 + 10 = 15$.

5.) On a particular day, an 8 foot pole casts a 6 foot shadow on level ground when the pole is inserted perpendicular to the ground. At the same time, an identical 8-foot pole also standing perpendicular to level ground casts a five foot shadow on a hill with an angle of elevation $\theta < 45^{\circ}$ to level ground. If $\sin(\theta) = \frac{a}{b}$ where a and b are integers with no common factors greater than 1, find $a + b$.



See the diagram. We can use the law of cosines to solve for the length AD, knowing $AB = 8$ and $BD = 5$, and $\cos(A) = \frac{4}{5}$. If $AD = x$, then $5^2 = 8^2 + x^2 - 2(8)(x)\left(\frac{4}{5}\right)$, which in standard form gives the quadratic $5x^2 - 64x + 195 = 0$, which factors to make $(x - 5)(5x - 39) = 0$, giving possible lengths of 5 and 7.8. If $AD = 5$, then our angle will be larger than 45°, so $AD = 7.8$. We can solve for AE using similarity: $\frac{AE}{8} = \frac{7.8}{10}$, so $AE = 6.24$, making $EB = 1.76$. Since $m\angle EDB = m\angle DBC = \theta, \sin(\theta) = \frac{1.76}{5} = \frac{44}{125}$, so $a + b = 44 + 125 = 169$.

6.) The line $x + 3y = 9$ intersects a circle centered at the origin with radius 5 at two points, creating a chord with endpoints D in quadrant I and E in quadrant II. If point F is placed on the circle such that $DE = EF$, then the line containing the points D and F has equation $Ax + By = C$, where $A > 0$ and $A, B,$ and C are integers that share no common factors greater than 1. Find $A + B + C$.



Refer to the diagram. We can solve for the points of intersection by solving the system $x^2 + y^2 = 25$ and $y = -\frac{1}{3}x + 3$. Using substitution, we get $x^2 + \left(-\frac{1}{3}x + 3\right)^2 = 25$ which simplifies to $5x^2 - 9x - 72 = 0$. This factors into $(5x - 24)(x + 3) = 0$, giving points $(-3,4)$ (the point in quadrant II) and $(4.8,1.4)$ (the point in quadrant I). The chord $\overline{EF}$ will be a reflection of $\overline{ED}$ over the diameter containing the origin. Therefore the chord $\overline{DF}$ will be the base of an isosceles triangle with the height represented by the chord with slope $-\frac{4}{3}$. This means that the line containing points D and F will have slope $\frac{3}{4}$, so its equation is $y = \frac{3}{4}(x - 4.8) + 1.4$, which in standard form becomes $15x - 20y = 44$, so $A + B + C = 15 - 20 + 44 = 39$.