FAIRFIELD COUNTY MATH LEAGUE 2021-2022

Match 1 Round 1
Arithmetic: Percents

1.) $\{6,4,6\}$

2.) $\{80,85,84\}$

3.) $\{15000,5000,6000\}$

Note: All solutions provided are for form A only. All forms of a problem have similar solution methods.

1.) How many natural numbers less than 100 produce a natural number when increased by $\{40\%, 25\%, 33\frac{1}{3}\%\}$ and then again by $\left\{33\frac{1}{3}\%, 16\frac{1}{6}\%, 15\%\right\}$?

Solution:
Increasing a number n by 40% is equivalent to multiplying it by $\frac{7}{5}$ and increasing it by $33\frac{1}{3}\%$ is equivalent to multiplying it by $\frac{4}{3}$. The result is the product $\left(\frac{7}{5}\right)\left(\frac{4}{3}\right)n = \frac{28}{15}n$, which will be a whole number if n is divisible by 15. There are 6 such natural numbers less than 100.

2.) One item in a store is discounted by 70% and then by an additional 20%. Another item undergoes a price increase of $\{20,60,50\}\%$ and then a decrease of $x\%$. If the total percent discount on both items ends up being the same, find the value of x.

Solution:
Discounting a quantity n by 70% and then by an additional 20% is equivalent by multiplying it by .3 and .8 respectively. This gives a result of $.24x$, or a total percent discount of 86%. Increasing the price by 20% then discounting by $x\%$ makes $1.2\left(1 - \frac{x}{100}\right)n$. Setting $.24 = 1.2\left(1 - \frac{x}{100}\right)$ yields $x = 80$.

3.) For positive numbers A, B, and C, it is known that $A\%$ of B is equal to the difference between 30% of A and 40% of B. If $A\%$ of C is {13,11,14} and $B\%$ of C is {6,7,9}, find the value of the product ABC.

Solution:

From the first relationship, we know that $\frac{A}{100} * B = \frac{3}{10}A - \frac{4}{10}B$, which is equivalent to $AB = 30A - 40B$. Similarly from the next two statements we know $AC = 1300$ and $BC = 600$. Multiplying the first equation by C gives $ABC = 30AC - 40BC$, so $ABC = 30(1300) - 40(600) = 15000$.

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Match 1 Round 2
Algebra 1: Equations

1.) $\{84,98,93\}$

2.) $\{271,451,631\}$

3.) $\{126,70,30\}$

Note: All solutions provided are for form A only. All forms of a problem have similar solution methods.

1.) If $x = 4$ is the solution to the equation $\{38,52,47\}x + 192 = kx + 8$ where k is a constant, find the value of k.

Solution:
Setting $x = 4$ gives an equation of $38(4) + 192 = 4k + 8 \rightarrow 152 + 192 = 4k + 8$ which yields $k = 84$.

2.) If p and q are positive integers such that $11p - \{3,5,7\}q = \{2,1,4\}$, and $q < 1000$, find the largest possible value of p.

Solution:
We note by inspection that $(1,3)$ satisfies $11p - 3q = 2$. We then note that values of p and q will increase with p increasing by 3 and q increasing by 11. For example, the next ordered pair that satisfies the equation is $(4,14)$. Since q has a restriction, we can find how many increases of 11 q can have using $3 + 11n < 1000$. The largest integer n that satisfies this is $n = 90$, so the largest value of p is $1 + 90 * 3 = 271$.

3.) If the equation $\sqrt{x^2 + ax} = x + \{14,10,6\}$ has a positive integer solution for x, find the second-largest possible value of a.

Solution:
Squaring both sides produces $x^2 + ax = x^2 + 28x + 196$, which produces a solution for x of $x = \frac{196}{a-28}$. The second-largest possible value of a would

produce the second-smallest possible positive integer value of x, which is $x = 2$. Therefore $a - 28 = 98$, and $a = 126$.

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Match 1 Round 3 Geometry: Triangles & Quadrilaterals

1.) $\{120,135,150\}$

2.) $\{120,80,40\}$

3.) $\{30,36,24\}$

Note: All solutions provided are for form A only. All forms of a problem have similar solution methods.

1.) A quadrilateral has angles whose measures in degrees form an arithmetic sequence. If the second largest angle measures $\{100,105,110\}$ degrees, what is the measure of the largest angle in degrees?

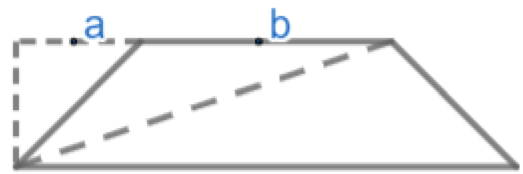
<u>Solution:</u>
Letting n be the measure of the second largest angle in degrees. We then get $n - 2d + n - d + n + n + d = 360$, or $4n - 2d = 360$. Since $n = 100$, this gives us $400 - 2d = 360$, or $d = 20$, so the measure of the largest angle is $100 + 20 = 120$ degrees.

2.) A rhombus has one diagonal that is three times the length of its other diagonal and an area of $\{540,240,60\}$. What is its perimeter?

<u>Solution:</u>
Diagonals of a rhombus bisect each other and are perpendicular to each other. Let the smaller diagonal have length $2d$ and the larger diagonal have length $6d$. Then the area is $\frac{1}{2}(2d)(6d) = 540$, so $d = 3\sqrt{10}$. One side length s therefore satisfies $d^2 + (3d)^2 = s^2$, so $s = d\sqrt{10} = 30$. Therefore the perimeter of the rhombus is $4(30) = 120$.

3.) An isosceles trapezoid has a height of $\{3\sqrt{2}, 2\sqrt{3}, 2\sqrt{2}\}$ units and a diagonal length of $\{2\sqrt{17}, 2\sqrt{30}, 4\sqrt{5}\}$ units. What is the area of the trapezoid in square units?

Solution:



Refer to the diagram. Since this is an isosceles trapezoid, it follows that $(a+b)^2 + \left(3\sqrt{2}\right)^2 = \left(2\sqrt{17}\right)^2$, so $a + b = \sqrt{50}$. For an isosceles trapezoid, the midsegment length is exactly $a + b$, so the area is $\left(3\sqrt{2}\right)\left(\sqrt{50}\right) = 30$. Note that this problem can also be modeled assuming $a = 0$ and treating the isosceles trapezoid as a rectangle.

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Match 1 Round 4 Algebra 2: Simultaneous Equations

1.) $\{5,8,1\}$

2.) $\{25,36,100\}$

3.) $\{35,42,47\}$

Note: All solutions provided are for form A only. All forms of a problem have similar solution methods.

1.) If (p,p) where p is a constant is the solution for (x,y) to the system $\begin{cases} Ax+2y=\{16,21,4\} \\ 3x-Ay=\{9,19,1\} \end{cases}$ where A is a constant, find the value of p.

Solution:

Replacing x and y with p and adding the equations yields $3p+2p=25$, so $p=5$.

2.) Consider the system $\begin{cases} \{2,3,9\}x+my=2m \\ (2m+1)x+5y=\{2m+6,2m+5,2m+1\} \end{cases}$ where m is a constant. If the system has no solutions for (x,y), find the value of $4m^2$.

Solution:

The determinant of the system is $(2)(5)-m(2m+1)$, so simplifying and setting equal to 0 yields $2m^2+m-10=0$, giving solutions of $m=2$ and $m=-\frac{5}{2}$. However we need to check these as these could yield either no solution or infinite solutions. Checking $m=2$ gives the system $\begin{cases} 2x+2y=4 \\ 5x+5y=10 \end{cases}$, which has infinite solutions. Checking $m=-\frac{5}{2}$ gives the system $\begin{cases} 2x-\frac{5}{2}y=-5 \\ -4x+5y=1 \end{cases}$, which has no solutions. Therefore our desired answer is $4\left(-\frac{5}{2}\right)^2 = 25$.

3.) Three particular real numbers have a sum of $\{5\sqrt{3}, 6\sqrt{2}, 3\sqrt{15}\}$. The sum of the three products of two of the three numbers is $\{20,15,44\}$. Find the sum of the squares of the numbers.

<u>Solution:</u>
Let the three numbers be represented by x, y, and z. We have it that $x + y + z = 5\sqrt{3}$. This means that $(x + y + z)^2 = 75$. Expanding the binomial square gives $x^2 + y^2 + z^2 + 2(xy + yz + xz) = 75$, which means $x^2 + y^2 + z^2 = 75 - 2(20) = 35$.

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Match 1 Round 5
Precalculus: Right Triangle Trigonometry

1.) $\{16,26,52\}$

2.) $\{21,18,28\}$

3.) $\{48,60,120\}$

Note: All solutions provided are for form A only. All forms of a problem have similar solution methods.

1.) Consider right triangle TRI with right angle R. If $\sin(T) = TI = \left\{\frac{2}{3}, \frac{3}{4}, \frac{5}{6}\right\}$ then $TR = \frac{x\sqrt{y}}{z}$ where x and z have no common factors greater than 1 and y has no perfect square factors greater than 1. Find $x + y + z$.

Solution:

Since $\sin(T) = \frac{RI}{\frac{2}{3}} = \frac{2}{3}$, then $RI = \frac{4}{9}$ and $TR = \sqrt{\frac{4}{9} - \frac{16}{81}} = \frac{2\sqrt{5}}{9}$, so $x + y + z = 2 + 5 + 9 = 16$.

2.) A right triangle has the property that the tangent of one of its acute angles is $\{75\%, 80\%, 40\%\}$ larger than its sine. If the area of the triangle is $\{18\sqrt{33}, 20\sqrt{14}, 80\sqrt{6}\}$, find the length of the hypotenuse of the triangle.

Solution:

If $\tan(A) = \frac{7}{4}\sin(A)$, then $\cos(A) = \frac{4}{7}$. This means we can model the hypotenuse as $7k$, one leg as $4k$, and then by Pythagorean theorem the last leg would be $\sqrt{33}k$. Using the area, $\frac{1}{2}(4k)\left(\sqrt{33}k\right) = 2\sqrt{33}k^2 = 18\sqrt{33}$, so $k = 3$, making the hypotenuse length $7(3) = 21$.

3.) In right triangle ABC with right angle C, the sum of $\sin(A)$ and $\cos(A)$ is $\left\{\frac{2\sqrt{3}}{3}, \frac{2\sqrt{10}}{5}, \frac{\sqrt{42}}{6}\right\}$. If the triangle has an area of $\{4,9,5\}$ square units, find the square of the length of the hypotenuse.

Solution:

Given $\sin(A) + \cos(A) = \frac{2\sqrt{3}}{3}$, we know $(\sin(A) + \cos\,(A))^2 = \sin^2(A) + 2\sin(A)\cos(A) + \cos^2(A) = 1 + 2\sin(A)\cos(A) = \frac{4}{3}$, so $\sin(A)\cos(A) = \frac{1}{6}$. Since $\sin(A)\cos(A) = \frac{a}{c} * \frac{b}{c} = \frac{ab}{c^2}$, and $\frac{1}{2}ab = 4$, we know $\frac{8}{c^2} = \frac{1}{6}$, so $c^2 = 48$.

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Match 1 Round 6 Miscellaneous: Coordinate Geometry

1.) {13,11,9}

2.) {14,22,32}

3.) {57,22,71}

Note: All solutions provided are for form A only. All forms of a problem have similar solution methods.

1.) If a particular line has an x-intercept of $(-1,0)$ and the line passes through $\{(10,66),(8,48),(12,52)\}$, and its equation in standard form is $Ax + By = C$ where A, B, and C are integers with no common factors greater than 1 and $A > 0$, find the value of $A - B - C$.

Solution:
In slope-intercept form $y = mx + b$, we know that $m = b$ since the line contains the point $(-1,0)$. Using the point $(10,66)$, we get $66 = 10m + m$, so $m = b = 6$, so $y = 6x + 6$. In standard form the equation becomes $6x - y = -6$, so $A - B - C = 6 - (-1) - (-6) = 13$.

2.) The circle with equation $(x-3)^2 + (y+k)^2 = r^2$ has a diameter with endpoints $(2,-5)$ and $(a,\{-11,-13,-15\})$. Find the value of $k - a + r^2$.

Solution:
We know that the center of the circle $(3,-k)$ must be the midpoint of the diameter endpoints $(2,-5)$ and $(a,-11)$. This means that $a = 4$ and $k = 8$. We can use the point$(2,-5)$ and the equation to find $(2-3)^2 + (-5+8)^2 = 10 = r^2$. Therefore $k - a + r^2 = 8 - 4 + 10 = 14$.

3.) An isosceles triangle is graphed on the coordinate plane such that its base has coordinates $\{(-3,-3),(-1,-1),(-6,-6)\}$ and $\{(-13,-13),(-9,-9),(-12,-12)\}$. Its vertex has coordinates (p,q). If the triangle has area $\{10\sqrt{7}, 8\sqrt{3}, 6\sqrt{10}\}$, find the value of pq.

Solution:
The line segment that serves as the height of the triangle given the base coordinates is perpendicular to the base and intersects the base at its midpoint. The base has length $10\sqrt{2}$ and midpoint $(-8,-8)$. We can find the height of triangle using $\frac{1}{2}\left(10\sqrt{2}\right)h = 10\sqrt{7}$, so $h = \sqrt{14}$. The slope of the base is 1 so the height has slope -1, which means the height of the triangle is the hypotenuse of an isosceles right triangle with hypotenuse length $\sqrt{14}$. This makes the length of each leg $\sqrt{7}$, so the coordinates of the vertex of the isosceles triangle $\left(-8+\sqrt{7}, -8-\sqrt{7}\right)$ (or vice-versa), making the product $64 - 7 = 57$.

FAIRFIELD COUNTY MATH LEAGUE 2021-2022 Match 1 Team Round

1.) 458	4.) 90
2.) 7	5.) 1618
3.) 57	6.) 5

1.) How many positive integers n have the property that decreasing n by 25% or increasing n by 20% results in an integer less than or equal to 1000?

Solution:

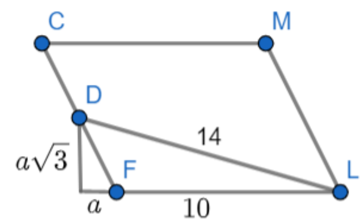
From the first condition, we get $\frac{3}{4}n < 1000$, so $n < 1333.\overline{3}$. So any positive integer value of n divisible by 4 and less than $1333.\overline{3}$ will work. The largest positive integer that meets these requirements is 1332, making 333 numbers. By the second condition, we get $\frac{6}{5}n < 1000$, or $n < 833.\overline{3}$, so any positive integer value of n divisible by 5 and less than $833.\overline{3}$ will work. The largest positive integer that meets these requirements is 830, making 166 total numbers. To count all numbers meeting the first or second condition, we apply the addition rule, noting that there are floor $\left(\frac{833.\overline{3}}{20}\right) = 41$ numbers that meet both conditions, giving a total of $333 + 166 - 41 = 458$ total numbers.

2.) If the equation $\frac{ax+2}{x+1} = \frac{cx-\frac{6}{x+1}}{bx-3}$ has infinite solutions for x and both a and b are nonzero, how many different integer pairs (a, b) exist?

Solution:

Transforming this to a polynomial through cross-multiplication yields $abx^2 + (2b - 3a)x - 6 = cx^2 + cx - 6$. From this, we know there will be infinite solutions if $ab = 2b - 3a$. We can solve for b in terms of a to get $b = \frac{3a}{2-a}$. We can see that as a increases in magnitude, b approaches -3, so we can look for values that work but stop once b's value is within one unit of -3. This gives the following ordered pairs: $(-4,-2), (-1,-1), (1,3), (3,-9),\ (4,-6),\ (5,-5)$, and $(8,-4)$, or 7 total ordered pairs.

3.) Consider parallelogram $FCML$ with midpoint D on $\overline{FC}$. If $m\angle F = 120^{\circ}$, $FL = 10$ and $LD = 14$, then the area of the parallelogram can be written as $a\sqrt{b}$ where b has no perfect square factors larger than 1. Find the value of $a - b$.



Solution:

Refer to the diagram. Because the measure of angle F is 120 degrees, we can create a 30-60-90 triangle outside of the parallelogram to make a larger right triangle with LD as the length of the hypotenuse. This means $(10 + a)^2 + (a\sqrt{3})^2 = 196$, which gives an equation of $4a^2 + 20a + 100 = 196$, or $a^2 + 5a - 24 = 0$, which yields $a = 3$ or $a = -8$, which is extraneous. Therefore the height of the parallelogram is $6\sqrt{3}$, so the area is $10 * 6\sqrt{3} = 60\sqrt{3}$, so $a - b = 60 - 3 = 57$.

4.) Given the system $\begin{cases} Ax + 5y = 60 \\ \frac{5}{x} + \frac{6}{y} = 2 \end{cases}$, the set of all constants A such that the system has no real solutions for (x, y) is bounded below by m and above by n, where m and n are positive integers. Find the value of $2n - 3m$.

Solution:

Using the first equation, we get $5y = 60 - Ax$. Substituting into the second equation gives $\frac{5}{x} + \frac{30}{60-Ax} = 2$. Multiplying each term by both denominator expressions yields the quadratic $5(60 - Ax) + 30x = 2x(60 - Ax)$, which simplifies to $2Ax^2 - (5A + 90)x + 300 = 0$. This will have no real solutions when the discriminant is less than zero, so setting up $(5A + 90)^2 - 4(300)(2A) <= 0$ expands to make $25A^2 - 1500A + 8100 < 0$, or $A^2 - 60A + 324 < 0$, which factors to make $(A - 6)(A - 54) < 0$. Therefore this system has no solutions for $6 < A < 54$, and so $2n - 3m = 2(54) - 3(6) = 90$.

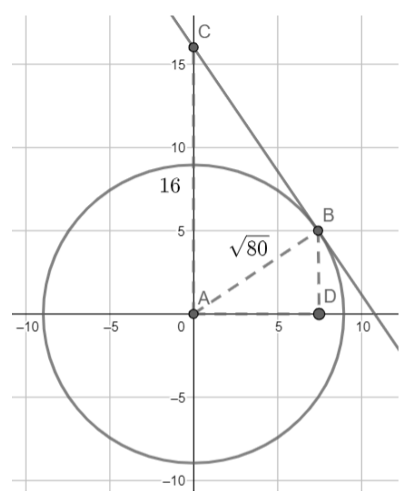
5.) In right triangle ABC with larger acute angle A the ratio of $\tan(A)$ to $\sin(A)$ is the square of the ratio of $\sin(A)$ to $\cos(A)$. Find the value of $1000\sec(A)$ rounded to the nearest whole number.

<u>Solution:</u>

From the problem we know $\frac{\tan(A)}{\sin(A)} = \left(\frac{\sin(A)}{\cos(A)}\right)^2$, which simplifies to $\sec(A) = \tan^2(A)$. Using identities, $\sec(A) = \sec^2(A) - 1$, so $\sec^2(A) - \sec(A) - 1 = 0$. Solving the quadratic gives $\sec(A) = \frac{1+\sqrt{5}}{2} \approx 1.618$, so $1000\sec(A)$ rounded to the nearest whole number is 1618.

6.) A line with a y-intercept of $(0,16)$ lies tangent to the circle $x^2 + y^2 = 80$ in quadrant 1. The line intersects the circle at the point (a, b). Find the value of b.

<u>Solution:</u>



Refer to the diagram. We see that triangle ABC is similar to triangle BDA. Therefore $\frac{AC}{AB} = \frac{AB}{BD}$. Using our values, we get $\frac{16}{\sqrt{80}} = \frac{\sqrt{80}}{x}$, so $x = \frac{80}{16} = 5$.