FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 6 Round 1 Geometry: Lines and Angles

Note: Figures not necessarily
Drawn to scale

1.) __________{30, 15, 75} degrees___

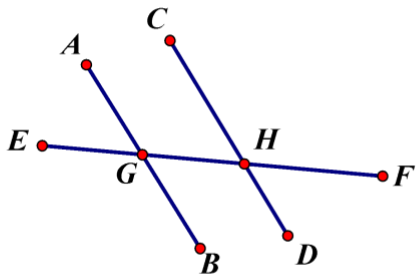
2.) ________{67, 63, 59}__degrees____

3.) ___________{8, 4, 2}_________

1.) What is the degree measure of the acute angle formed by the intersection of the lines $\{y = \sqrt{3}(x-2) + 1$, $y - 4 = x - 5$, $x + y = 3\}$ and $3y - 3 = \sqrt{3}(x-2)$?

Solution:
Both lines pass through (2,1) . The slope of $y = \sqrt{3}(x-2) + 1 = \sqrt{3}$, so it makes a 60 degree angle with the horizontal. The slope of= $3y - 3 = \sqrt{3}(x-2) = \frac{\sqrt{3}}{3}$, so it makes a 30 degree angle with the horizontal. The acute angle formed by the intersection of the lines is 60 - 30 = 30 degrees.

2.) In the figure below, segment AB is parallel to segment CD. The lines are cut by transversal line EF, which intersects line segment AB at G and segment CD at H. There is a number *x* such that the measure of angle AGE is {$(\frac{5}{2}x - 23)$, $(\frac{5}{2}x - 42)$, $(\frac{5}{2}x - 61)$ } degrees and the measure of angle GHD is $(\frac{2}{3}x + 89)$degrees. Find the measure of angle FHD.



<u>Solution:</u>

Angle AGE and angle BGH are vertical angles, and angle BGH and angle GHD are same side interior angles, so the sum of AGE and GHD is 180 degrees.

$(\frac{5}{2}x - 23)+(\frac{2}{3}x + 89)= 180$

$(\frac{15}{6}x - 23)+(\frac{4}{6}x + 89)= 180$

$\frac{19}{6}x + 66= 180$

$\frac{19}{6}x$=114, x=36. Angle FHD is congruent to angle AGE, so angle FHD measures $\frac{5}{2} * 36 - 23$= 67 degrees.

3.) Rhombus WXYZ has W at {(-4,3), (-8,6), (-6,8) } and X at (0,0). Y is in quadrant I and the slope of line XY is { $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{3}$} . The slope of XZ is {$a + b\sqrt{5}$, $a + b\sqrt{10}$, $a + b\sqrt{10}$ } for some integer values of a and b. Find a+b.

Solution:

Each side of the rhombus measures 5, since $\sqrt{(-4)^2 + 3^2}$= 5. To find point Y, $\sqrt{x^2 + (0.5x)^2}$=5, so $\frac{5}{4}x^2$=25, $x = 2\sqrt{5}$,y=$\sqrt{5}$. Slope of WY is $\frac{\sqrt{5}-3}{2\sqrt{5}+4}$.

Diagonals of a rhombus are perpendicular to each other, so the slope of XZ is $\frac{2\sqrt{5}+4}{3-\sqrt{5}}=\frac{(2\sqrt{5}+4)(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{22+10\sqrt{5}}{4}=\frac{11+5\sqrt{5}}{2}$. $\frac{11}{2}+\frac{5}{2}$=8

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Match 6 Round 2 Algebra: Literal Equations

1.) ______ {1,3,4}______________

2.) _______{ 14, 6, 36}__________

3.) ______ {1,3,2}_____________

1.) If the equation $\frac{1}{2}(x+3y-8)-z=3z-2(4x-y)$ is solved for z in terms of x and y, then z=ax+by+c. Find {a+b+c, a+b-c, a+b-2c}.

Solution:

$$\frac{1}{2}(x+3y-8)-z=3z-2(4x-y)$$
$$(\frac{1}{2}x+\frac{3}{2}y-4)-z=3z-8x+2y$$
$$(\frac{1}{2}x+\frac{3}{2}y-4)+8x-2y=4z$$
$$(\frac{17}{2}x-\frac{1}{2}y-\ 4\ =\ 4z$$
$$(\frac{17}{8}x-\frac{1}{8}y-1)=z$$

$\frac{17}{8}$-$\frac{1}{8}$-1=2-1=1.

2.)_ If the equation { $x=\frac{y}{2}+\frac{1}{y}$, $x=y+\frac{1}{y}$, $x=\frac{y}{4}+\frac{1}{y}$ is solved for y, $y=\frac{ax\pm\sqrt{bx^2+c}}{2}$ for some values of a, b, and c. Find a + b - c.

Solution:

$x=\frac{y}{2}+\frac{1}{y}$

$2yx=y^2+2$

$y^2-2yx+2=0.$

$y=\frac{2x\pm\sqrt{4x^2-8}}{2}$ so 2 + 4 - (-8) = 14

3.) Suppose that $x > 0$. When the equation

$\{ 2xy(x^2 + 1) - 3x^2 = (2x^2 + 1)(2x^2 - 1) - x^2y - y, \; 4xy(x^2 + 1) - 15x^2 = (4x^2 + 1)(4x^2 - 1) - x^2y - y, 3xy(x^2 + 1) - 8x^2 = (3x^2 + 1)(3x^2 - 1) - x^2y - y \}$ is solved for y, the result is $y = ax + b$, for some constants a and b. Find $a + b$.

Solution:

$$2xy(x^2 + 1) - 3x^2 = (2x^2 + 1)(2x^2 - 1) - x^2y - y$$
$$2xy(x^2 + 1) - 3x^2 = (2x^2 + 1)(2x^2 - 1) - y(x^2 + 1)$$
$$2xy(x^2 + 1) + y(x^2 + 1) = (2x^2 + 1)(2x^2 - 1) + 3x^2$$

$$y(2x + 1)(x^2 + 1) = (4x^4 - 1) + 3x^2$$
$$y(2x + 1)(x^2 + 1) = (4x^2 - 1)(x^2 + 1)$$
$$y(2x + 1)(x^2 + 1) = (2x + 1)(2x - 1)(x^2 + 1)$$

$y = \frac{(x^2+1)(2x-1)(2x+1)}{(2x+1)(x^2+1)}$=2x-1

a+b = 2+(-1) = 1

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Match 6 Round 3 Geometry: Solids and Volumes

1.)______{60, 15,135}___________

2.)_______{ 4860, 1440, 180 }____

3.) ____{48, 192, 108} _______

1.) A cone has horizontal base and its vertex lies vertically above the center of the base. The cone has height {8, 4, 12}, and its volume is {96π, 12π, 324π}. The lateral area of the cone (the surface area not including the base) is Aπ. What is A?

Solution:

$\frac{1}{3}\pi r^2 \quad * 8 = 96\pi,\ r^2 = \frac{3*96}{8}$=36, r=6. Slant height is $\sqrt{6^2+8^2}$=10 inches. Lateral area is π*6*10 = 60π. A = 60.

2.) A sphere of radius {9, 6, 3} cm is inscribed in a cube. The volume that is outside the sphere but inside the cube is $a - b\pi$ cm^3. What is a - b?

Solution:

The radius of the sphere must be half the side of the cube, so the cube has side 18 cm. The volume of the cube is 18^3 = 5832 cubic cm. The volume of the sphere is $\frac{4}{3}\pi * 9^3 = 972\pi$. $5832 - 972 = 4860$.

3.) A pyramid has a square base, and the base is horizontal. The height of the pyramid is {4, 8, 6}. A horizontal plane cuts the pyramid into two parts such that the volume of the top part is ½ of the volume of the bottom part. If the height of this plane above the base is *k*, then $k = a - b\sqrt[3]{c}$,
where *a* and *b* are rational numbers and *c* is an integer that is not divisible by the cube of any prime number. Find the product of *a*, *b*, and *c*.

<u>Solution:</u>
The plane cuts the pyramid into a smaller pyramid and another solid, and the volume of the smaller pyramid must be one third of the volume of the original pyramid. These two pyramids are similar, and the volume scale factor between them is 1/3. Hence the linear scale factor between them must be $\frac{1}{\sqrt[3]{3}}$. So the height of the small pyramid is $\frac{4}{\sqrt[3]{3}}$. Therefore, the height of the cutting plane is $4 - \frac{4}{\sqrt[3]{3}} = 4 - \frac{4}{3}\sqrt[3]{9}$. Hence, the answer to the question is

$4 \cdot \frac{4}{3} \cdot 9 = 48$.

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Match 6 Round 4 Radical Expressions and Equations

1. ________ {8, 11, 13}___________

2. ________{448, 384, 576}

3. ________{71, 67, 63}_____________

1.)_ For how many integer values of K is $\{2+\sqrt{K+3}-K, 4+\sqrt{K+3}-K, 6+\sqrt{K+3}-K\}$ a positive number?

Solution:

K cannot be less than -3 in order to take the square root. We need $\sqrt{K+3}$ + 2 - K < 0, so K-2>$\sqrt{K+3}$, so k^2 - 4k + 4 < k+3, so k^2 - 5k + 1 < 0. True for k=-3, -2, -1, 0, 1,2,3,4, so there are 8 such integers.

2.) Suppose that $a_0, a_1, a_2,$... is a sequence of numbers such that a_0 = x, $a_1 = \sqrt{a_0}$, $a_2 = \sqrt{a_1}$, $a_3 = \sqrt{a_2}$, and so on. If a_6 = {2^7, 2^6, 2^9], then x = 2^n. Find n.

Solution:

$a_6 = 2^7$, $a_5 = 2^{14}$, $a_4 = 2^{28}$, $a_3 = 2^{56}$, $a_2 = 2^{112}$, $a_1 = 2^{224}$, $a_0 = 2^{448}$,

3.) For how many integers n with $0 \le n \le \{10000,9000,8000\}$ is $\sqrt{2n+1}$ rational?

Solution:

Since $\sqrt{20001}$= 141.4, we can go up to 2n+1 = 19881, 2n=19880, n=9940.

2n+1 must be odd and a perfect square, so 2n+1 could be 1, 9,25,49,... where n=0, 4, 12, 24, 40, 60... 9940. This sequence is 2k(k-1). For what value of k is $2k(k-1)=9940$? $2k^2 - 2k - 9940 = 0$, $k^2 - k - 4970=0$, so by quadratic formula $k = 71$

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Match 6 Round 5 Polynomials and Advanced Factoring

1. _______{8, 12 , 16 }_____________

2. _______{75, 131, 35}___________

3. ________{7, 44, 159 }___________

1.) Let $f(x) = x^3 + Ax + B$ and suppose that $f(1) = \{3,5,7\}$ and $f(2) = \{15,20,25\}$. Find $|A| + |B|$.

Solution:
3 = 1^3 + A*1 +B and 15 = 2^3 + A*2 +B, so A+B=2 and 2A+B=7. Then A=5, B=-3
|5| + |-3| = 8

2.

$$\{x^4 - 5x^3 + 17x^2 - 45x + K,$$
$$x^4 - 5x^3 + 24x^2 - 80x + K,$$
$$x^4 - 5x^3 + 12x^2 - 20x + K\}$$

factors to

$$\{(x^2+9)(x^2+Bx+C), (x^2+16)(x^2+Bx+C), (x^2+4)(x^2+Bx+C)\}.$$

Find K+B+C.

Solution:
$(x^2+9)(x^2+Bx\ +C)$ = $x^4 + Bx^3\ +\ (9+C)x^2\ +\ 9Bx\ +\ 9C$
9B = -45, so B =- 5. 9+C = 17, so C=8. K = 9C=9*8=72.
-5+8+72 = 75

3.

A quartic polynomial $x^4 + Ax^3 + Bx^2 + Cx + D$, where A, B, C, D are integers, has {2+i and 1-2i , 3+i and 1-3i, 4+i and 1-4i} as two of its zeros, where $i = \sqrt{-1}$. Find A+B+C+D.

Solution:

Since the coefficients are real, if 2+i is a zero, so is 2-i. If 1-2i is a zero, so is 1+2i. The quadratic with zeros 2+i and 2-i has $\frac{-b}{a}$=4,$\frac{c}{a}$=5 so it is $x^2 - 4x + 5$, since the leading coefficient must be 1 to give x^4 (-1 will just require the second polynomial to also have leading coefficient -1, and the negatives cancel out).

The quadratic with zeros 1-2i and 1+2i has $\frac{-b}{a}$=2, $\frac{c}{a}$=5, so it is $x^2 -2x+5$

Multiply $(x^2 - 4x + 5)(x^2 -2x+5) = x^4 -4x^3 + 5x^2 - 4x^3 + 8x^2 - 20x + 5x^2 - 10x + 25 =$

$x^4 -6x^3 +18x^2 -30x+25$.

-6+18-30+25 =7

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Match 6 Round 6
Counting and Probability

1.) _______{6,5,6}______

2.) ________{ 240, 80, 560}_________

3.) _____{31, 37, 9}___________

1.) $_NC_R$ denotes the number of combinations of N objects taken R at a time. For how many of the {10,13,15} integer values of R, $0 \leq R \leq$ {9,12,14}, is {$_9C_R$,$_{12}C_R$,$_{14}C_R$} divisible by {9,12,14} ?

Solution:
The values $_9C_0$, ... $_9C_9$ are 1, 9, 36, 84, 126, 126, 84, 36, 9, 1, and 9, 36, 126, 126, 36, and 9 are divisible by 9, so there are 6.

2.) The {12,10,14} members of a club consist of {6,5,7} married couples. A subset of 4 club members will be selected to represent the club at a conference. In how many ways can this be done if no person and his/her spouse may both be selected?

Solution:
There are 12 choices for the first person, 10 for the second, 8 for the third, 6 for the fourth. Multiply 12*10*8*6 and order doesn't matter so divide by 4! to get 240.

3) {Four nickels and six dimes, Three nickels and six dimes, Four nickels and five dimes} are placed in a bag, and five coins are drawn from the bag at random without replacement. The probability that the value of the coins is at least 40 cents is $\left\{\frac{A}{42}, \frac{A}{42}, \frac{A}{14}\right\}$. Find *A*.

<u>Solution:</u>

It could be 5 dimes, 4 dimes and one nickel, or 3 dimes and 2 nickels.

$$\frac{(6^C5)(4^C0)}{(10^C5)}+\frac{(6^C4)(4^C1)}{(10^C5)}+\frac{(6^C3)(4^C2)}{(10^C5)}=$$

$$\frac{6}{252}+\frac{15*4}{252}+\frac{20*6}{252}=\frac{186}{252}=\frac{31}{42}$$

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Match 6 Team Round

1.) ________186_____ 4.) _______6_____________

2.) ________167_____ 5.) _____ 23_________

3.) _______16_______6.) _______31744____

1.) A, B, C, and D are the interior angles of a convex quadrilateral ABCD. The measure of the supplement of angle D is six degrees more than the measure of angle B. Find the sum of angles A and C.

Solution:

A + B + C + D = 360º

180º - D = B + 6

B + D = 174º ==> A + C = 186º

2.) If $k = \sqrt[3]{3 + \sqrt[3]{3 + \sqrt[3]{3 + \sqrt[3]{3 + \dots}}}}$ and k is real, what is $100k$ rounded to the nearest integer?

Solution:

Note that $k = \sqrt[3]{3 + k}$. So $k^3 = 3 + k$. Using technology to solve this equation we get $k = 1.6717$. So the integer closest to $100k$ is 167.

3.) A regular tetrahedron has volume $\frac{16\sqrt{2}}{3}$. The surface area is $K\sqrt{3}$ for some value of K. Find K.

Solution:

$\frac{16\sqrt{2}}{3} = \frac{a^3}{6\sqrt{2}}$ *where a = edge of tetrahedron*

a = 4

Area of an equilateral triangle is $\frac{a^2\sqrt{3}}{4}$

The surface area of the tetrahedron is 4* $\frac{a^2\sqrt{3}}{4}$ =16√3 so K = 16.

4.) $x^3 + Ax^2 + Bx - 6$ factors into three binomials with integer coefficients. What is the absolute value of the sum of all possible values of *A* ?

Solution:

Find all possibilities and find the opposite of the sum of the roots.

Possibilities are: (x+1)(x+2)(x-3) : A = -(-1-2+3) = 0
(x-1)(x+2)(x+3) : A = -(1-2-3)=4
(x+1)(x-2)(x+3) : A = -(-1+2-3)=2
(x-1)(x-2)(x-3) : A = -(1+2+3) = -6
(x-1)(x+1)(x+6) : A = -(1-1-6) = 6
(x+1)(x+1)(x-6) A = -(-1-1+6) = -4
(x-1)(x-1)(x-6)= A = -(1+1+6) = -8
|0 + 4 + 2 + -6 + 6 + -4 + -8| = 6

5) When three standard six-sided dice with sides labeled 1 through 6 are rolled, the probability that the sum is 12 or 13 is $\frac{K}{108}$. Find *K*.

Solution:

To get 12: One way to get 4,4,4 Three ways each to get 2,5,5 and 3,3,6
Six ways each to get 1,5,6 and 2,4,6 and 3,4,5. Total is 25.
To get 13: Three ways each to get 6,6,1 and 5,5,3 and 4,4,5
Six ways each to get 3,4,6 and 2,5,6. Total is 21. Probability is $\frac{46}{216} = \frac{23}{108}$. K = 23.

6) A softball player has a probability 0.4 of getting a hit in any at bat. She comes to bat 5 times in one game, and the results of her at-bats are independent. The probability that she gets at least 3 hits is *K*.
Find 100,000**K*.

Solution:

$({}_5C_3)(0.4)^3(0.6)^2 + (({}_5C_4)(0.4)^4(0.6)) + ({}_5C_5)(0.4)^5(0.6)^0$

$^{=}$ 10*(0.064)(0.36)+5*(0.0256)(0.6) + 1*(0.01024)=
.2304 + .0768 + 0.01024 = .31744.
100000*(0.31744) = 31744