FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 5 Round 1
Arithmetic: Fractions & Exponents

1.) {21, 17, 16}

2.) {67, 60, 47}

3.) {1, 9, 16}

Note: Solutions are for form A only. All forms have similar solution methods.

1.) Let an “h-cube” be a cube whose volume is half a cubic centimeter. A number of h-cubes will now be piled up, one on top of the other. What is the smallest number of h-cubes that are needed in order to make a pile whose height is more than {16,13,12} centimeters?

Setting up an inequality to solve for the number of cubes (x) required to surpass a height of 16 cm gives the equation $\sqrt[3]{\frac{1}{2}} * x > 16$, so $x > 16\sqrt[3]{2}$, making the least possible integer that satisfies the inequality 21.

2.) A fraction is written with n as its numerator and $n + \{6,10,15\}$ as its denominator. For how many integers n, with $1 \leq n \leq 100$, is the fraction reducible?

Considering the fraction $\frac{n}{n+6}$, this quantity will be reducible if and only if n and $n + 6$ share a common factor greater than 1. This common factor would have to be a common factor of both n and 6, so any n with 2 or 3 as a factor would count. We can therefore apply the addition rule to find the answer by computing (multiples of 2) + (multiples of 3) – (multiples of 6), which comes out to $50 + 33 - 16 = 67$.

3.) If $\frac{(2^{4x})(3^{6y})}{(8^{2y})(9^{x})\left(\frac{1}{9^{-x}}\right)} = \left\{\frac{3}{2}, \frac{8}{27}, \frac{81}{16}\right\}$, find the value of $16x^2 - 48xy + 36y^2$.

Converting every power in the fraction expression to base 2 or 3 gives $\frac{2^{4x} * 3^{6y}}{2^{6y} * 3^{2x} * 3^{2x}}$, which can be written as $\left(\frac{2}{3}\right)^{4x-6y}$. Setting this equal to $\frac{3}{2}$ means $4x - 6y = -1$. Since $16x^2 - 48xy + 36y^2 = (4x - 6y)^2$, the solution is $(-1)^2 = 1$.

FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 5 Round 2 Algebra 1: Fraction Expressions and Equations

1.) {15, 10, 12}

2.) {25, 9, 16}

3.) {6, 15, 36}

Note: Solutions are for form A only. All forms have similar solution methods.

1.) If *N* liters of orange juice are shared equally between {6,8,9} people, then each person gets $\{\frac{7}{8}, \frac{7}{12}, \frac{2}{3}\}$ of a liter more than if $N - 2$ liters are shared equally between $\{8, 12, 15\}$ people. Find *N*.

Setting up an equation in terms of the number of liters each person receives makes $\frac{N}{6} = \frac{7}{8} + \frac{N-2}{8}$, which is equivalent to $4N = 21 + 3(N - 2)$, which leads to the solution $N = 15$.

2.) The equation $\frac{1}{n} - \frac{\{10,6,8\}}{n^2} + \frac{a}{n^3} = 0$, where a is a nonzero constant, has only one solution for n. Find the value of a.

Since $n \neq 0$, we can multiply the equation by n^3 to yield $n^2 - 10n + a = 0$. This will have only one solution for n when the trinomial is a perfect square, making $a = 25$. (You can also set up $10^2 - 4a = 0$ to solve directly.)

3.) Let *k* be that constant such that one of the solutions of the equation $\frac{2x+3}{x+k} = \frac{x-1}{3x+4}$ is $x = \{2.4, 1.5, 1.2\}$. Find the other solution for x.

Since $x = 2.4$ is a solution, we can substitute it into the equation to yield $\frac{7.8}{2.4+k} = \frac{1.4}{11.2} = \frac{1}{8}$, which can be solved for k using $k = 8 * 7.8 - 2.4 = 60$. This gives us the equation $\frac{2x+3}{x+60} = \frac{x-1}{3x+4}$, which by cross-multiplication gives $6x^2 + 17x + 12 = x^2 + 59x - 60$, or $5x^2 - 42x + 72 = 0$, which has solutions of $x = \frac{12}{5}$ and $x = 6$, which is our desired answer.

FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 5 Round 3
Geometry: Circles

1.) {52, 48, 44}

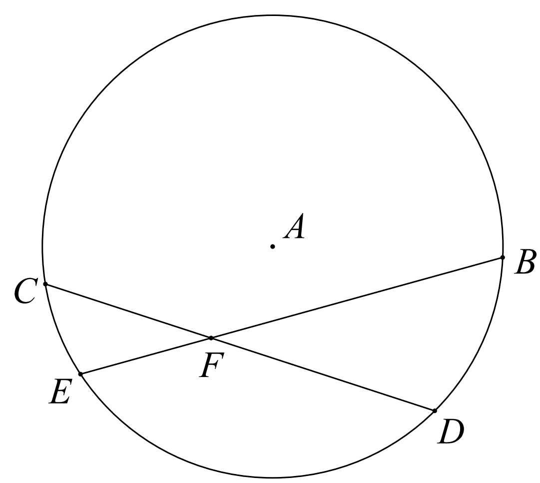
2.) {198, 126, 70}

3.) {45, 200, 60}

Note: Solutions are for form A only. All forms have similar solution methods.

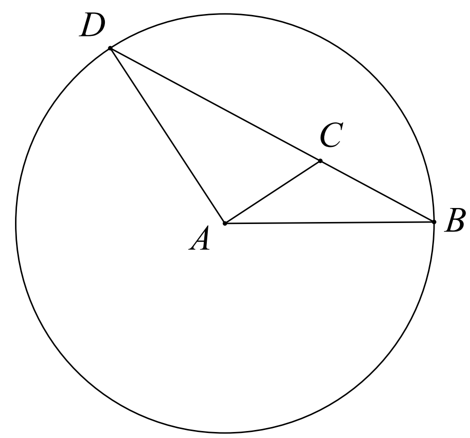
1.) See the diagram. The chords $\overline{CD}$ and $\overline{BE}$ intersect at point F. If $m\angle CFB = 144°$ and the measure of arc EC is $\{20,24,28\}$ degrees, find the degree measure of arc BD.

We know $m\angle CFE = 36°$ from the given information, and then using the fact that $m\angle CFE = \frac{m\widehat{EC}+m\widehat{BD}}{2}$, it follows that $36 = \frac{20+x}{2}$, so $x = 52$.



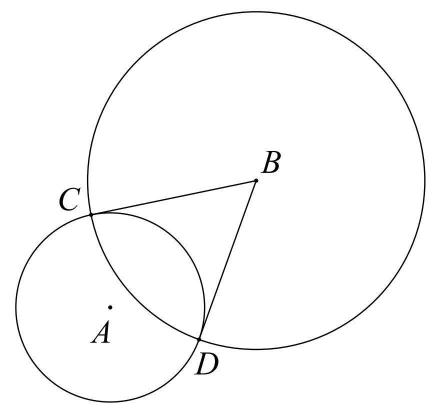
2.) In the diagram, A is the center of the circle, $\angle DAB \cong \angle ACB$, $BC = \{9,7,5\}$, $BD = \{22,18,14\}$, and the area of the circle is $m\pi$. Find the value of m.

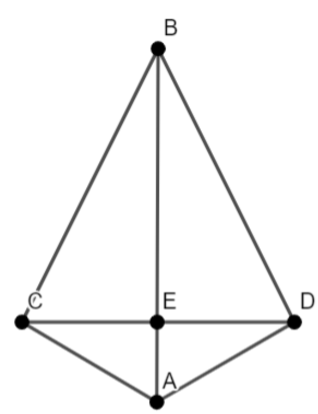
This can be approached multiple ways but perhaps the easiest is to use the fact that triangle ABD is similar to triangle CAB, since $\angle D \cong \angle B$ (as triangle ABD is isosceles). Therefore we can set up $\frac{BC}{AB} = \frac{AD}{BD}$, or $\frac{9}{x} = \frac{x}{22}$, so $x^2 = 198$. As this is the square of the radius, it is our desired value of m.



3.) See the diagram. The circle with center B intersects the circle with center A at points C and D such that $\overline{BC}$ and $\overline{BD}$ are tangent to circle A. If the area of circle A is $\{30\pi, 50\pi, 40\pi\}$, $CD = \{6\sqrt{2}, 4\sqrt{10}, 4\sqrt{6}\}$ and the area of circle B is $k\pi$, find the value of k.

See the diagram of the kite $BCAD$, with $\overline{CD}$ drawn in with $\overline{CD}$ and $\overline{AB}$ intersecting at point E. Because $\overline{BC}$ and $\overline{BD}$ are tangents, it follows that triangle BCA is a right triangle, and therefore triangle BCE is similar to trinagle CAE. Therefore $\frac{BC}{CE} = \frac{CA}{AE}$, and using the information given we know $\frac{BC}{3\sqrt{2}} = \frac{\sqrt{30}}{\sqrt{(\sqrt{30})^2 - (3\sqrt{2})^2}}$, which means $(BC)^2 = 45$, which is our desired result.





FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 5 Round 4
Algebra 2: Quadratic Equations and Complex Numbers

1.) {4, 2, 6}

2.) {4, 4, 3}

3.) {1000,1250,1300}

Note: Solutions are for form A only. All forms have similar solution methods.

1.) Let $f(z) = z^2 - \{4,2,6\}z$. There exists a complex number w such that $f(w) = f(i)$. Let $w = a + bi$ where a and b are nonzero real numbers. Find the value of a.

One way to approach this problem is to write $f(z) = z(z-4)$, so $f(i) = i(i-4) = -i(4-i) = (4-i)(-i) = (4-i)(4-i-4)$, so $w = 4-i$. Another way is to note that since the vertex is located at $z = 2 + 0i$, the difference between w and $2+0i$ must be the same as the difference between $2 + 0i$ and i, which is $2 - i$. This gives $w = 2 - i + 2 + 0i = 4 - i$.

2.) Let k be the complex constant such that the discriminant of the equation $z^2 - 3iz + k = 0$ is $\{6+i, 2+5i, 4-i\}$. If $k = a + bi$, where a and b are real, find $|a+b|$.

We know that k is the product of the zeros of the equation. By the quadratic formula, we know the zeros take the form $\frac{3i+\sqrt{6+i}}{2}$ and $\frac{3i-\sqrt{6+i}}{2}$, making the product $\frac{(3i)^2-(6+i)}{4}$, or $-\frac{15}{4} - \frac{1}{4}i$, so $\left|-\frac{15}{4} - \frac{1}{4}\right| = 4$.

3.) The equation $z^2 + az + b = 0$, where a and b are complex constants, has two zeros, z_1 and z_2, Both zeros are natural number powers of $\{1+2i, 2+$

$i, 1+3i\}$. If $|z_1| = \{5\sqrt{5}, 5\sqrt{5}, 10\}$ and $|z_2| = \{25{,}25{,}10\sqrt{10}\}$, find $|a|^2$.

Since $|1+2i| = \sqrt{5}$, it follows that $z_1 = (1+2i)^3 = -11-2i$ and $z_2 = (1+2i)^4 = -7-24i$. Since a is the opposite of the sum of the zeros, $a = -(-11-2i-7-24i) = 18+26i$. Therefore $|a|^2 = 18^2 + 26^2 = 1000$.

FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 5 Round 5 Trig and Advanced Topics: Trigonometric Equations

1.) {21, 17, 19}

2.) {14, 12, 10}

3.) {7, 9, 6}

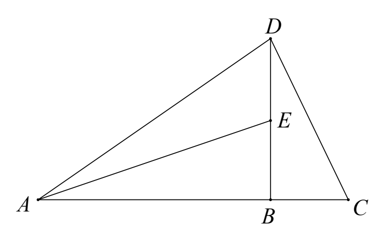
Note: Solutions are for form A only. All forms have similar solution methods.

1.) How many solutions x are there to the equation $\cos\left(\{10,8,9\}x + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, with $0 \leq x \leq 2\pi$?

Letting $10x + \frac{\pi}{6} = \frac{\pi}{6} + 2n\pi$ gives $x = \frac{1}{5}n\pi$, giving answers of $n \in \{0,1,2,3,4,5,6,7,8,9,10\}$. Letting $10x + \frac{\pi}{6} = \frac{11\pi}{6} + 2n\pi$ gives $x = \frac{\pi}{6} + \frac{1}{5}n\pi$, giving answers of $n \in \{0,1,2,3,4,5,6,7,8,9\}$. In total this gives 21 total solutions.

2.) In the diagram, $\overline{BD}$ is perpendicular to $\overline{AC}$. $AD = \{4,5,6\}$ and $DC = 2$. Let $m\angle DAE = m\angle EAB = m\angle BDC = \theta$, where $0 < \theta < 45°$. Find θ, rounding your answer to the nearest degree. (Do not include a unit.)

From the information and the diagram we can set up $\sin(2\theta) = \frac{DB}{4}$ and $\cos(\theta) = \frac{DB}{2}$, giving $2\sin(2\theta) = \cos(\theta)$, or $4\sin(\theta)\cos(\theta) = \cos(\theta)$, which gives $\sin(\theta) = \frac{1}{4}$, or $\theta \approx 14$ degrees.



3.) Given that

$$\{(k+3)\sec^2 x - 2k = 3\tan^2 x + 3\sec x\tan x,$$
$$(k+1)\sec^2 x - (2k-4) = 3\tan^2 x + 3\sec x\tan x,$$
$$(k+4)\sec^2 x - (2k+2) = 3\tan^2 x + 3\sec x\tan x\},$$

where k is a constant, the product of all possible values of $\sin x$ is $-\frac{4}{11}$. Find the value of k.

Multiplying the equation by $\cos^2(x)$ transforms it to $(k+3) - 2k\cos^2(x) = 3\sin^2(x) + 3\sin(x)$. Substituting in $1 - \sin^2(x)$ for $\cos^2(x)$, collecting like terms and writing in standard form in terms of $\sin(x)$ gives $(2k-3)\sin^2(x) - 3\sin(x) + 3 - k = 0$, or dividing by $2k-3$, $\sin^2(x) - \frac{3}{2k-3}\sin(x) + \frac{3-k}{2k-3} = 0$. This means $-\frac{4}{11} = \frac{3-k}{2k-3}$, which can be solved to get $k = 7$.

FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 5 Round 6
Miscellaneous: Sequences and Series

1.) $\{673, 504, 1008\}$

2.) {22, 78, 112}

3.) {81, 64, 125}

Note: Solutions are for form A only. All forms have similar solution methods.

1.) The first three terms of an arithmetic sequence are $\{2.2, 2.8, 3.4, \ldots;\ 2.6, 3.4, 4.2, \ldots;\ 3.8, 4.2, 4.6, \ldots\}$. How many terms of the sequence are less than 2021 and are integers?

Note that the first integer term of the sequence will be 4, and that each additional integer term will be three more than the integer term before it. Therefore we can set up $4 + 3(n-1) < 2021$, which gives $n < 673\frac{1}{3}$, so $n = 673$.

2.) Let $a_0, a_1, a_2, \ldots$ be an arithmetic sequence with common difference d. Let $b_0, b_1, b_2, \ldots$ be a geometric sequence with common ratio $\{1.2, 1.6, 1.8\}$. If $a_0 = b_0$ and $a_2 = b_2$, what is $100\frac{d}{a_0}$?

Note that $a_2 = a_0 + 2d$ and $b_2 = b_0(1.2)^2 = 1.44a_0$. Setting up $a_0 + 2d = 1.44a_0$ gives $2d = .44a_0$, so $100\frac{d}{a_0} = 22$.

3.) An infinite geometric series has the property that the first term minus the second term is $\{225, 196, 245\}$ and the first term minus the third term is $\{75, 49, 147\}$. What is the sum of the series?

From the problem we can set up $a_0 - a_0r = 225$ and $a_0 - a_0r^2 = 75$.

From this we know $\frac{75}{225} = \frac{a_0(1-r^2)}{a_0(1-r)} = 1 + r$, so $r = -\frac{2}{3}$. We also know that $a_0 = \frac{225}{1+\frac{2}{3}} = 135$. Therefore the sum of the series is $\frac{135}{1-(-\frac{2}{3})} = 81$.

Team Round
FAIRFIELD COUNTY MATH LEAGUE 2020-21 Match 5 Team Round

1.) 20
2.) 21
3.) 11
4.) 63
5.) 30
6.) 630

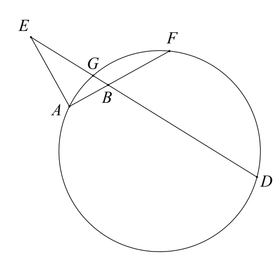
1.) Let $x_1, x_2, x_3 \dots$ be a sequence of numbers such that $x_1 = 3$ and, for $n \geq 1$, $x_{n+1} = \frac{1}{1+x_n}$. Find the reciprocal of the product $x_1 * x_2 * \dots * x_8$.

We can use the recursion to see $x_2 = \frac{3}{4}$, $x_3 = \frac{4}{7}$, and the pattern continues with each numerator being the prior denominator and each denominator being the sum of the prior numerator and denominator. Therefore the product cancels all numerators and denominators, leaving $\frac{3}{60}$, whose reciprocal is 20.

2.) Consider the equation $x + \frac{1}{a+\frac{1}{x}} = b$, where a and b are constants. If the solutions for x have the same absolute value and $a + b = 5$, find the value of $a^2 + b^2$.

Multiplying every term by $a + \frac{1}{x}$ gives $ax + 2 = ab + \frac{b}{x}$, and then multiplying by x and collecting terms gives $ax^2 + (2 - ab)x - b = 0$. In order for the solutions to have the same absolute value, $2 - ab = 0$, so $ab = 2$. Note that $(a + b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2(2) = 25$, so $a^2 + b^2 = 21$.

3.) See the diagram (not necessarily drawn to scale). Chords $\overline{DG}$ and $\overline{AF}$ intersect at point B and $\overline{AE}$ is perpendicular to $\overline{AF}$. $BG = 2$, $BD = 10$, $AE = BF$, and $GE = 7$. Find AF.



We know $(BG)(BD) = (AB)(BF)$, so $(AB)(BF) = 20$. Also, from the right triangle and the information given, $(AB)^2 + (BF)^2 = 81$. $(AF)^2 = (AB + BF)^2 = (AB)^2 + 2(AB)(BF) + (BF)^2 = 81 + 2(20) = 121$, so $AF = 11$.

4.) If a and b are positive real numbers such that $\left|\frac{5+ai}{b+3i}\right| = 1$ and $|a + bi| = \sqrt{65}$, find $2ab$.

From the first equation we know $25 + a^2 = b^2 + 9$, so $b^2 - a^2 = 16$. From the second equation we know $b^2 + a^2 = 65$. Adding the equations gives $2b^2 = 81$, so

$b = \frac{9}{\sqrt{2}}$. Solving for a gives $a = \frac{7}{\sqrt{2}}$. Therefore $2ab = 63$.

5.) If $\theta = \tan^{-1}(\frac{1}{5})$ is a solution to the equation $\tan(\theta + A) = \frac{4}{7}$, find $27\sec^2(A)$.

Expanding the tangent expression gives $\frac{\tan(\theta)+\tan(A)}{1-\tan(\theta)\tan(A)} = \frac{4}{7}$. Allowing $\theta = \tan^{-1}\left(\frac{1}{5}\right)$ gives $\frac{\frac{1}{5}+\tan(A)}{1-\frac{1}{5}\tan(A)} = \frac{4}{7}$. This can be solved to yield $\tan(A) = \frac{1}{3}$. Then $\sec^2(A) = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$, and $27\sec^2(A) = 30$.

6.) Consider a geometric series and an arithmetic sequence, where the first term of the geometric series is equal to the first term of the arithmetic sequence, and the 50th term of the geometric series is equal to the 50th term of the arithmetic sequence. The common difference of the arithmetic sequence is -10. If the geometric series converges to a value that is $\frac{9}{7}$ times its first term, find the sum of the first 49 terms of the geometric series.

Let a_0 be the first term of both the series and the sequence (since they are equal) and let r be the common ratio of the geometric series. We know $a_0 - 490 = a_0 r^{49}$. This means $a_0(1 - r^{49}) = 490$. We also know that $\frac{a_0}{1-r} = \frac{9}{7}a_0$, so $\frac{1}{1-r} = \frac{9}{7}$. The sum of the first 49 terms of the geometric series is $\frac{a_0(1-r^{49})}{1-r} = (490)\left(\frac{9}{7}\right) = 630$.