FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 4 Round 1 Arithmetic: Basic Statistics

1.) _________{2,1,1}_________

2.) ______ {153,156,150}______

3.) ________{20,50,80}________

1) What is the positive difference between the arithmetic mean and the median of the set {prime numbers between {55, 65, 75} and 100}? Round to the nearest whole number if necessary.

Solution:

Arithmetic mean: $\frac{59+61+67+71+73+79+83+89+97}{9}=\frac{679}{9}=75.4$

Median: 73

Difference is 75.4-73 = 2.4, round to 2.

2) When a set of data is listed in order of size, the upper quartile is the median of the upper half of the data set, and the lower quartile is the median of the lower half of the data set. What is the sum of the upper quartile, the median, and the lower quartile of the set of numbers {multiples of {6,8,4} between 1 and 99}?

Solution:

Set is 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96.

Median is $\frac{48+54}{2}=51$

Lower Quartile is $\frac{24+30}{2}=27$

Upper Quartile is $\frac{72+78}{2}=75$

Sum is 27+51+75 = 153

3) You have a collection of quarters and dimes. If you add 30 nickels to your collection, you will have {150,180,210 } coins altogether and the mean value per coin will decrease by $\left\{\frac{3}{2},\frac{5}{3},\frac{5}{3}\right\}$ cents. How many quarters do you have?

Solution:

Let Q = # of quarters, D=# of dimes. Q+D+30 = 150, so Q+D=120.

$\frac{25Q+10D}{120}-\frac{25Q+10D+150}{150}=1.5$

Multiply by 600

5(25Q+10D)-4(25Q+10D+150)=900

25Q+10D-600=900, 25Q+10D=1500.

Solve the system 25Q+10D=1500, Q+D=120

so 25Q+10D=1500, 10Q+10D=1200. Subtract the equations to get 15Q=300, so Q=20.

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Match 4 Round 2
Algebra 1: Quadratic Equations

1.) _______{10,2,1}_______

2.)_____{49,50,46}________

3.) {256, 2400, 2940}

1) If R is the positive root of {9,4,2}({2,3,4}x^2 - x) = {20,1,3} and T is the negative root of {9,4,2}({2,3,4}x^2 - x) = {20,1,3}, what is {10R+4T, 7R+9T, 8R+10T}?

<u>Solution:</u>

$18x^2 - 9x = 20$, $18x^2 - 9x - 20 = 0$, $(6x+5)(3x-4)=0$, $x=-\frac{5}{6}, \frac{4}{3}$

$10(\frac{4}{3})+4(\frac{-5}{6})=\frac{80}{6} - \frac{20}{6} = 10$

2) A quadratic equation has solutions $\{\frac{4+5\sqrt{2}}{3}$ and $\frac{4-5\sqrt{2}}{3}$, $\frac{3+5\sqrt{2}}{3}$ and $\frac{3-5\sqrt{2}}{3}$, $\frac{1+5\sqrt{2}}{3}$ and $\frac{1-5\sqrt{2}}{3}\}$. The equation can be expressed as $Ax^2 + Bx + C = 0$, where A, B, and C are relatively prime integers and $A > 0$. Find $|A + B + C|$.

Solution:

Sum of $\frac{4+5\sqrt{2}}{3}$and $\frac{4-5\sqrt{2}}{3}$ is $\frac{8}{3}$

Product of $\frac{4+5\sqrt{2}}{3}$and $\frac{4-5\sqrt{2}}{3}$is $\frac{16-50}{9} = \frac{-34}{9}$

$x^2 - \frac{8}{3}x+\frac{-34}{9}=0$

$9x^2 - 24x - 34=0$

9-24-34=-49

3) Find the product of all values of k for which the equation $\{(k-8)x^2 + kx + (k-3)$, $(k-15)x^2 +kx +(k-8)$, $(k-21)x^2 + kx +(k-5)\} = 0$ has exactly one real solution for x.

Solution:

$k^2 - 4(k-8)(k-3)=0$, $k^2 - 4(k^2-11k+24)=0$, $-3k^2 + 44k-96=0$,

$3k^2 - 44k + 96=0$, $(3k-8)(k-12)=0$, $k=12$ or $\frac{8}{3}$. Product is $12*\frac{8}{3}=32$. However, if $k = 8$ there is also exactly one solution to the equation. So, the answer to the question is $32\cdot 8 = 256$.

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Match 4 Round 3
Geometry: Similarity
Note: Diagrams are not
Necessarily drawn to scale

1) ________{64, 80, 112} ________

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2.) ________{26,56,34}_________

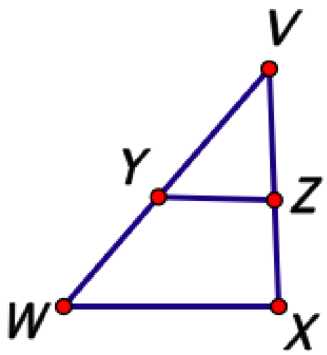
3.) _____{14, 126, 56}_________

1) The ratio of the areas of two regular octagons is {16:9, 25:9, 49:9}. One side of the smaller octagon measures 6 cm. Find the perimeter of the larger octagon.

<u>Solution:</u>
The ratio of the perimeters is the square root of the ratio of their sides, so the ratio of the perimeters is $\frac{4}{3}$. The perimeter of the smaller octagon is 6*8 = 48 cm, so $\frac{4}{3} = \frac{P}{48}$, P = 64 cm.

2) $\triangle VWX$ is a right triangle with the right angle at X. Point Y lies on segment VW and point Z lies on segment VX. Segment YZ is parallel to segment WX. WX = {5, 11, 7}, $VX = 3\sqrt{3}$, and $YZ = 4$. As a radical expression in simplest form, $VY = \frac{A\sqrt{B}}{C}$. Find $A + B + C$.

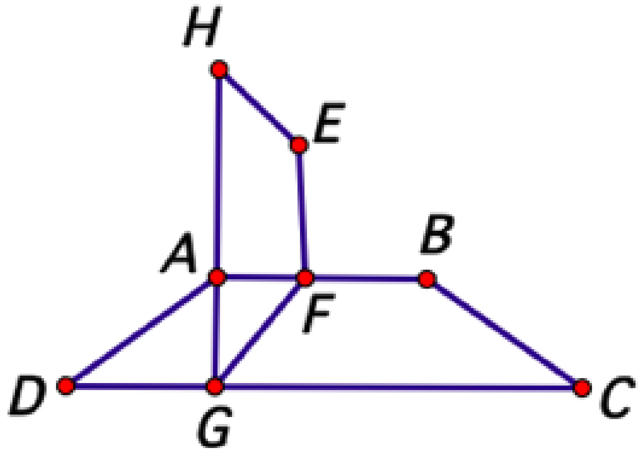
<u>Solution:</u>



Since ΔWVX is a right Δ, $WV^2 = (3\sqrt{3})^2 + 5^2 = 52$, $WV = 2\sqrt{13}$

$\frac{VY}{VW} = \frac{YZ}{WX}$, $\frac{VY}{2\sqrt{13}} = \frac{4}{5}$, $4 * 2\sqrt{13} = 5(VY)$, $5(VY) = 8\sqrt{13}$, $VY = \frac{8\sqrt{13}}{5}$. So the answer to the question is 8 + 13 +5 = 26.

3) In the diagram below, not necessarily drawn to scale, isosceles trapezoid ABCD is similar to trapezoid EFGH with segment AB parallel to segment CD. AB+CD={28,84,56} and 11(AB)=3(CD). HG ^ DC and AB ^ EF. BC={ $4\sqrt{5}, 12\sqrt{5}, 8\sqrt{5}$}. Find the area of EFGH.



<u>Solution:</u>

AB=$\frac{3}{11}$CD, so CD+$\frac{3}{11}$CD=28, $\frac{14}{11}$CD=28, CD=22. AB=28-22=6. Since ABCD is isosceles, DG=0.5(22-6)=8. Since BC=AD=$4\sqrt{5}$, $AG^2=(4\sqrt{5})^2-8^2$, so AG=4.

$\frac{AF}{AG}=\frac{AG}{DG}$, $\frac{AF}{4}=\frac{4}{8}$, AF=2. $\frac{HG}{DC}=\frac{AG}{DG}$, so $\frac{HG}{22}=\frac{2}{4}$, HG=11, and similarly EF=3.

$\frac{1}{2}$(11+3)*2=14.

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Match 4 Round 4
Algebra 2: Variation

1.) ___{9,6,2}___________

2.) __{5625, 2500, 36}____

3.) ________{11,17,14}____

1.) $(z-10)$ varies inversely with the square root of w. If z=8 when w=64, what is the value of z when w={256,16,4} ?

Solution:
z-10=$\frac{k}{\sqrt{w}}$, 8-10=$\frac{k}{\sqrt{64}}$, k=-16. z-10 = $\frac{-16}{\sqrt{256}}$=, so z-10=-1, z=9

2.) m varies inversely with n, and n^2 varies directly with p^3.
If p=25 when m=27000, what is p when m={8,27,15625} ?

Solution:
m=$\frac{k_1}{n}$, $n^2=k_2p^3$, so n =$k_3p^{1.5}$, m = $\frac{K}{p^{1.5}}$ 27000 =$\frac{K}{25^{1.5}}$,

K =125*27000=3375000
8 = $\frac{3375000}{p^{1.5}}$, $p^{1.5}$=421875, p = 5625.

3.) The electromagnetic force in notwens between two charged particles in Universe X varies jointly with the product of their charges in bmoluocs and inversely with the cube of the distance in retems between them.
Particles A and B are 2 retems apart. The electromagnetic force between them is {3600,6000,4800} notwens. Particle B has charge 200 bmoluocs. The force between particles A and C is 120 notwens, and particle C has charge 60 bmoluocs. The distance between particles A and C in retems in simplest radical form is M$\sqrt[3]{N}$. What is M+N ?

<u>Solution:</u>
Let a = charge on particle A and k = constant of proportionality and R = distance between A and C.

$3600 = \frac{kA(200)}{2^3}$, so kA = 144. $120 = \frac{kA(60)}{R^3}$, $120 = \frac{(144)(60)}{R^3}$,
120R^3= 144*60=21600, R^3 = 72, R=$\sqrt[3]{72}$ = $2\sqrt[3]{9}$. 2+9 = 11

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Match 4 Round 5 Trig Expressions and DeMoivre's Theorem

1.) _{119,144,169}_____

2.) _____{3,3,1}_____

3.) ___{73,103,137}__

1.) For how many integers N with $1 \le N \le \{300, 325, 350\}$ is $\cos N°$ positive?

Solution:

89 values from 1 through 89, and then 30 more from 271 through 300. 89+30 = 119.

2.) If $r \operatorname{cis} \theta$ means $r(\cos\theta + i \sin\theta)$, for how many integer values of A between $\{-10$ and 0, -5 and 5, 0 and $10\}$ is $\left(4 \operatorname{cis} \frac{\pi}{A}\right)\left(6 \operatorname{cis} \frac{\pi}{A+3}\right)$ a complex number whose real part is equal to zero?

Solution:

We need $\frac{1}{A} + \frac{1}{A+3}$ to be an odd multiple of 0.5. This is true for A=-6, -2, -1, and 3. So there are {3,3,1}

3.) Lines L_1 and L_2 both have positive slope and pass through the origin. Line L_1 also passes through the point {(7, 2), (8,3), (9,4)}. The angle between lines L_1 and L_2 is equal to the angle between line L_1 and the positive *x*-axis. The slope of line L_2 is $\frac{p}{q}$, where *p* and *q* are relatively prime positive integers. Find $p + q$.

Solution:

Call the angle between L_1 and the x-axis ø. The tangent of ø is $\frac{2}{7}$. The angle between L_1 and L_2 is also ø, so the angle between L_2 and the positive x-axis is 2ø. $\tan(2ø) = \frac{2\,tan(ø)}{1-tan^2(ø)}$. $\tan(2ø) = \frac{2*\frac{2}{7}}{1-(\frac{2}{7})^2} = \frac{\frac{4}{7}}{1-\frac{4}{49}} = \frac{\frac{4}{7}}{\frac{45}{49}} = \frac{28}{45}$. p+q = 73.

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Match 4 Round 6 Conics

1.) ____________{39,56,24}____________

2.)____________{3,4,5}________

3.)____________ {21,16,4}__________

1.) A line with a negative slope is tangent to the circle $x^2 + y^2 = 25$ at the point T. The line also passes through {A(8,0), A(9,0), A(7,0)}. Find the square of the length AT.

Solution:

The line is tangent to the circle at T. If O is the origin, the triangle OAT is a right triangle with the right angle at T. OT = 5 since it is a radius of the circle. OA = 8. $AT^2 + 5^2 = 8^2$, so $AT^2 = 8^2 - 5^2 = 39$

2.) A hyperbola has foci at ({3,4,5}$\sqrt{5}$, 0) and ($-${3,4,5}$\sqrt{5}$, 0) and one of its asymptotes has equation $y = 2x$. The hyperbola intersects the positive x-axis at (a, 0). Find a.

Solution:

If the intercepts are (a,0) and (-a,0), an asymptote has equation $y = \frac{b}{a}x = \frac{2}{1}x$ and $a^2+b^2 = c^2 = (3\sqrt{5})^2 = 45$. b=2a, from the asymptote equation, so $a^2 + (2a)^2 = 45$, $5a^2 = 45$, $a^2 = 9$, a=3.

3.) An ellipse has foci at $(2\sqrt{21}, 0)$ and $(-2\sqrt{21}, 0)$. It intersects the *y*-axis at (0, 4) and (0, −4). One intersection point of this ellipse with the circle centered at (0,0) with radius 6 has *y*-coordinate $\{\frac{16\sqrt{A}}{A}, \frac{A\sqrt{21}}{21}, \frac{A^2\sqrt{21}}{21}\}$. Find A.

<u>Solution:</u>
Find x-intercepts by $a^2 = (2\sqrt{21})^2+4^2$, $a= \pm 10$. x-intercepts are at (-10,0) and (10, 0), so the ellipse has equation $\frac{x^2}{100}+\frac{y^2}{16} =1$, so $4x^2 + 25y^2 = 400$. Multiply the equation of the circle by 4 to get $4x^2 + 4y^2 = 144$, subtract to get $21y^2 = 256$. $y=\sqrt{\frac{256}{21}}=\frac{16}{\sqrt{21}}=\frac{16\sqrt{21}}{21}$.
A=21.

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Answers:
1. 63
2. 3
3. 4
4. 10240
5. 25
6. 17

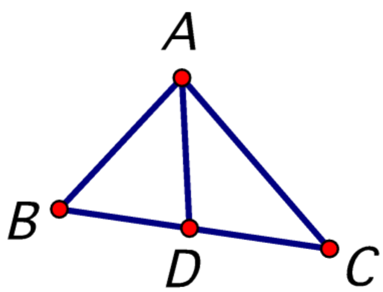
1) The geometric mean of a set of N numbers {a_1, a_2, ..., a_N} is defined to be $\sqrt[N]{a_1 a_2 a_3 \ldots a_N}$. A particular set of six numbers has geometric mean equal to 3. Five of the numbers are $\frac{7}{9}$, $\frac{1}{27}$, $\frac{1}{49}$, 81, and 243. What is the sixth number?

<u>Solution:</u>
We need $3^6 = \frac{7}{9} * \frac{1}{27} * \frac{1}{49} * 81 * 243 * a_N$,
$7*3^{-2}*3^{-3}*7^{-2}*3^4*3^5*a_N = 3^6$, so $a_N = \frac{3^6}{(7^{-1})(3^4)} = 7*9 = 63$

2) For how many integer values of *k* does the equation
$kx^2 + 3x + k = 25$ have two rational solutions?

<u>Solution:</u>
We need the discriminant $3^2 - 4(k(k-25)) = -4k^2 + 100k + 9$ to be positive. This will only be true for $0 \le k \le 25$. Then we need a perfect square. Set up a spreadsheet as k goes from 0 to 25 and check how many are perfect squares. For k=0 and k=25, the discriminant is 9. For k=11 and k=14, the discriminant is 625. But if k=0, you do not have a quadratic, so the only values for k are 11, 14, 25. There are three values of *k* for which the given equation has two rational solutions.

3.) In the diagram of ΔABC below, not necessarily drawn to scale, line AD bisects $\angle BAC$. AC=6x+3, DC=2x+7, BC=7x-3, AB=4x+2. Find the value of x.

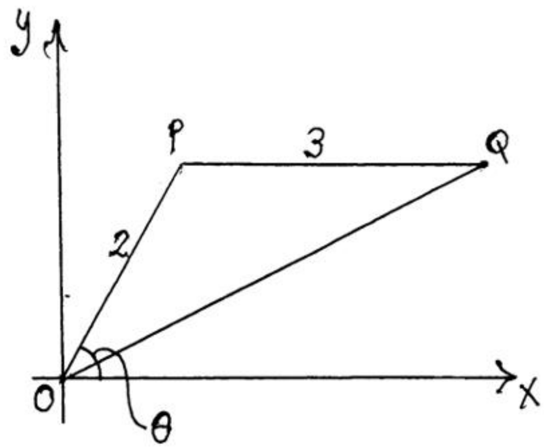


Solution:
By the angle bisector theorem proved using similar triangles, $\frac{AB}{BD}=\frac{AC}{CD}$.
BD=BC-CD=(7x-3)-(2x+7) = 5x-10.
$\frac{4x+2}{5x-10}=\frac{6x+3}{2x+7}$, (4x+2)(2x+7)=(5x-10)(6x+3),
$8x^2+32x+14=30x^2-45x-30$
$22x^2-77x-44=0$
$11(2x^2-7x-4)=0$
11(x-4)(2x+1)=0, x=4 or x=-0.5, but x=-0.5 gives negative side lengths, so x=4.

4.) The graph of a direct variation function of the form $y = kx^n$ passes through the points (64,8) and $(4,\frac{1}{128})$. What is $\frac{n}{k}$?

Solution:
$8=k*64^n$
$\frac{1}{128}=k*4^n$
Divide the two equations to get $1024=16^n$. $2^{10} = (2^4)^n$, so $n=\frac{5}{2}$.
$8=k*64^{\frac{5}{2}}$, $8= k*8^5$, $k = \frac{1}{8^4}$, $\frac{n}{k}=8^4 * \frac{5}{2}=2^{11}*5=2048*5=10240$.

5. In the diagram, line segment OP makes an acute angle θ with the x-axis and has length 2. Segment PQ is parallel to the x-axis and has length 3. The square of the length of segment OQ is $a + b\cos^2\frac{\theta}{2}$. Find $a + b$.



Solution:

Draw a vertical line from P to the x-axis that intersects the x-axis at R. Then cos ø = $\frac{OR}{2}$, so OR = 2 cos ø. Draw a horizontal line from P to the y-axis that intersects the y-axis at S, so sin ø = $\frac{OS}{2}$, so OS = 2 sin ø. OQ^2 = $(OR + 3)^2 + (OS)^2$ = (2 cos ø + 3)^2 + (2 sin ø)^2 = 4 cos^2 ø + 12 cos ø + 9 + 4 sin^2 ø = 13 + 12 cos ø.
cos^2 $(\frac{ø}{2})$ = $\frac{1+cosø}{2}$ so cos ø = 2cos^2 $(\frac{ø}{2})$-1. OQ^2 = 13 + 12 (2cos^2 $(\frac{ø}{2})$-1)) = 13 + 24 cos^2 $(\frac{ø}{2})$ - 12 = 1 + 24 cos^2 $(\frac{ø}{2})$. $a + b$ = 1+ 24 = 25.

6.) A circle has equation $x^2 - 6x + y^2 + 10y = 2$. A parabola has its vertex at the center of the circle. The focus of the parabola has the same *x*-coordinate as the *x*-coordinate of its vertex, and the *y*-coordinate of the focus is 3 greater than the *y*-coordinate of the vertex. The *y*-coordinate of either intersection point of the two curves has form $A + B\sqrt{2}$ for integers A and B. Find $B - A$.

Solution:
Find the center of the circle by
$x^2 - 6x + 9 + y^2 + 10y + 25 = 34+2$, $(x-3)^2 + (y+5)^2 = 36$, circle has center (3, -5), radius 6. This is the vertex of the parabola and the focus is 3 units above the vertex, so the parabola has form
$y = \frac{1}{4*3}(x-3)^2 - 5$, $y=\frac{1}{12}(x-3)^2 - 5$
Substitute $12(y+5) = (x-3)^2$ into the equation of the circle to get
$12(y+5) + (y+5)^2 = 36$, $12y+60+y^2 + 10y+25=36$, $y^2 +22y+49=0$,
$y=\frac{-22\pm\sqrt{484-196}}{2}=\frac{-22\pm\sqrt{288}}{2}=\frac{-22\pm12\sqrt{2}}{2}=-11 \pm 6\sqrt{2}$. Since the focus of the parabola is above the vertex, the parabola opens up, so the y-coordinate of the intersection point must be greater than -5, so take
$-11 + 6\sqrt{2}$ B-A=6-(-11)=17